

The Theories of Relativity

These Course Notes are a supplement to the treatment of Relativity in Chapter 37 of the PHY132H textbook, Randall D. Knight, Physics for Scientists and Engineers, 2nd ed. (Pearson Addison-Wesley, 2008). The text remains our primary reference for this discussion. Here we will explore some of the conceptual foundations of Relativity. We will also briefly introduce the General Theory of Relativity, which is not discussed in the textbook.

There are two different theories of Relativity which are due to Einstein:

1. The Special Theory of 1905. This theory considers observers who are in uniform relative motion.
2. The General Theory of 1916. This theory considers observers who are in any state of relative motion, including relative acceleration. This will turn out to also be a theory of gravitation.

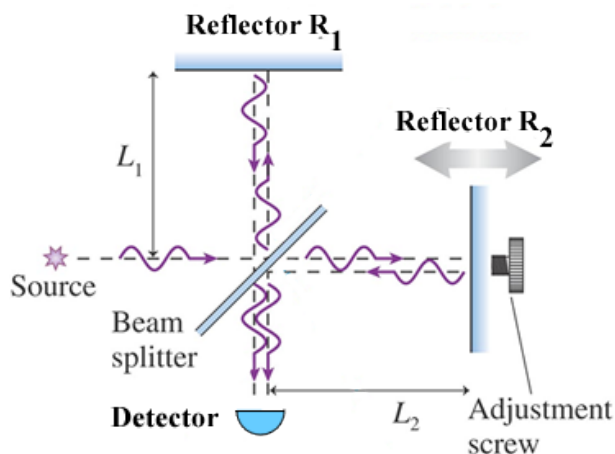
Below, discussion of Special Relativity will reference the section numbers of the text where this supplementary material is relevant. For the discussion of General Relativity, with the exception of the Principle of Equivalence sub-section of §13.3 the text does not discuss this topic so no section numbers can be given.

A: §37.3 - Einstein's Principle of Relativity

You will recall the discussion of the Michelson Interferometer in Chapter 22. The figure is slightly modified from Fig. 22.20 of the text.

A wave from the Source is split into two equal intensity parts by the Beam Splitter, sent to the two Reflectors, and returned to the Beam Splitter. Half of those returned rays are sent to the Detector.

Such an interferometer can be used with light waves, sound waves, or any other kind of wave. For a sound wave, the Source is a loudspeaker, the Beam Splitter can be just a piece of paper, the Reflectors can be glass plates, and the Detector is a microphone.



If the distances from the Beam Splitter to the Reflectors, L_1 and L_2 , are equal then the two waves will constructively interfere at the Detector provided the interferometer is stationary relative to the medium through which the waves are traveling.

Imagine we have constructed an interferometer for sound waves. Then if the arms of the interferometer have equal lengths and there is no wind blowing, we get constructive interference: the time it takes the two sound waves to travel from the Beam Splitter to the Reflectors and back to the Beam Splitter are equal. But if a constant wind of speed v is blowing from left to right then the time it takes the two waves to travel to the Reflectors and back will be unequal. If we call the speed of sound relative to the air c (343 m/s, not the speed of light!) then it is fairly simple to show that:

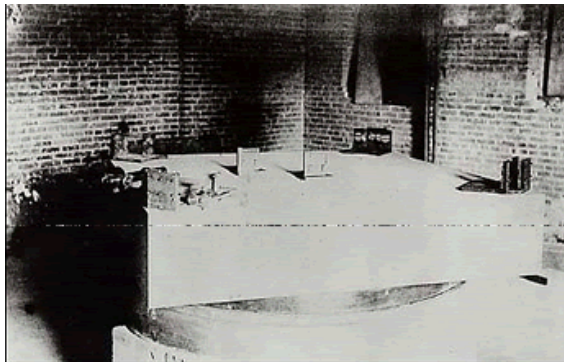
$$\frac{t_1}{t_2} = \sqrt{1 - v^2/c^2} \quad (\text{A.1})$$

Imagine that with the wind blowing we have adjusted the lengths L_1 and L_2 to compensate for the

different times it takes for the sound waves to travel to and from the two Reflectors so we get a maximum sound intensity at the Detector. Then if we slowly rotate the whole interferometer the carefully adjusted lengths will no longer give us constructive interference and, depending on the speed of the wind v , we could see the sound intensity go from maximum to minimum to maximum etc.

As discussed in the text, in the 19th century physicists thought about a medium for light which was called the *ether*. Presumably the ether is stationary relative to the fixed stars. But the surface of the Earth is not stationary relative to the fixed stars: it is rotating on its axis, in orbit around the Sun, and the whole solar system is moving towards the constellation Hercules. So on the surface of the Earth there will be an "ether wind" with a speed, depending on the time of day and season, as large as 3.0×10^4 m/s.

In the 1880's Michelson and Morley constructed a light interferometer to attempt to measure the speed of the ether wind relative to the surface of the Earth. The lengths of the arms of the interferometer were 10 m. To minimise vibration the whole apparatus was mounted on a marble block floating in a tub of mercury. They adjusted the interferometer for constructive interference, and then gently rotated the interferometer by 90 degrees. They calculated that they should easily see the combined beams going through fringe shifts in the interference pattern as they rotated the apparatus.



Michelson & Morley's 1887 interferometer
built in the basement of Western Reserve
Photo: Case Western Reserve Archive

When they did the experiment, they got no result: the interference pattern did not change! It was suggested that maybe the speed of the earth due to its rotation on its axis was canceling its speed due to its orbit around the sun. So they waited 12 hours and repeated the experiment. Again they got no result. It was suggested that the Earth's motion in orbit around the Sun cancelled the other motions. So they waited six months and tried the experiment again. And again they got no result. It was suggested that maybe the mass of the earth "dragged" the ether along with it. So they hauled the apparatus up on top of a mountain, hoping that the mountain would be sticking up into the ether that was not being dragged by the earth. And again they got no result. Thus, this attempt to measure the motion of the earth relative to the ether failed.

Lorentz was among many who were very puzzled by this result. He proposed that when an object was moving relative to the ether, its length along its direction of motion would be contracted by just the right amount needed to explain the experimental result. If the length of the object when it is at rest with respect to the ether is L_0 , then if it is moving at speed v through the ether its length L along its direction of motion relative to the ether is:

$$L = \sqrt{1 - v^2/c^2} L_0 \quad (\text{A.2})$$

Here c is the speed of light, not the speed of sound. So when the interferometer is oriented as in the figure and an ether wind is blowing from left to right, L_2 is contracted but L_1 is not.

When Einstein was 16, in 1895, he asked himself an interesting question:

"If I pursue a beam of light with the velocity c I should observe such a beam of light as a spatially oscillatory electromagnetic field at rest. However, there seems to be no such thing, whether on the basis of experience or according to [the theory of electricity and magnetism]. From the very beginning it appeared to me intuitively clear that, judged from the standpoint of such an observer, everything would have to happen according to the same laws as for an observer who, relative to the earth, was at rest. For how, otherwise,

should the first observer know, i.e., be able to determine, that he is in a state of uniform motion?" -- As later written by Einstein in "Autobiographical Notes", in Schilpp, ed., *Albert Einstein: Philosopher-Scientist* .

He continued to work on this question for 10 years with the mixture of concentration and determination that characterised much of his work. He published his answer in 1905:

"... light is always propagated in empty space with a definite velocity c which is independent of the state of [relative] motion of the emitting body The introduction of a 'luminiferous ether' will be superfluous inasmuch as the view here to be developed will not require an 'absolutely stationary space' provided with special properties." -- *Annalen Physik* 17 (1905).

Put another way, the speed of light is 3×10^8 m/s with respect to all observers.

Notice that this explains the null result of the Michelson-Morley experiment. However, although the evidence is not certain it seems quite likely that in 1905 Einstein was unaware of the experiment (cf. Gerald Holton, "Einstein, Michelson and the 'Crucial' Experiment," which has appeared in *Thematic Origins of Scientific Thought* , pg. 261. and also in *Isis* 60 , 1969, pg. 133).

A Flash animation of the Michelson-Morley Experiment by Wan-Ching Hui at the University of Virginia is available via the blue button to the right. It will open in a separate tab/window.



B: §37.6 - Time Dilation

When we discussed the Doppler effect (§20.7) we deferred discussion of the Doppler Effect for Light Waves sub-section. We are now ready to discuss this topic.

For mechanical waves, we made a distinction between two different cases:

1. The observer is stationary relative to the medium and the source is moving relative to the medium. This is described by Equation 20.39.
2. The source is stationary relative to the medium and the observer is moving relative to the medium. This is described by Equation 20.40.

For light waves there is no medium, so the distinction between these cases disappears: only the relative motion of the source and observer count.

For the reference frame of the observer the textbook's Figure 20.26 applies. We will discuss what Pablo measures.

Just as for the non-relativistic case, the wavelength is greater for Pablo than for an observer stationary relative to the source: the wave is "stretched". However there is another factor for the relativistic case: the period T of the source is greater than the period T_0 of the source relative to an observer stationary relative to the source by the factor γ defined by Equation 37.22. This second effect, due to relativistic time dilation, is what we were not able to discuss when we first talked about the Doppler effect.

A homework problem will let you show that the relativistic Doppler effect, Equation 20.41, is correct.

C: §37.7 - Length Contraction

C.1 *The Spacetime Interval* Sub-Section

This sub-section of the text begins by discussing two students with rotated reference frames. Here I will extend the analogy: The Parable of the Surveyors.

Once upon a time there was a kingdom in which all positions were measured relative to the town square of the capitol.

This kingdom had a sort of strange religion that dictated that all North-South distances were to be measured in sacred units of *feet*; East-West distances were measured in everyday units of *meters*. Despite this religious requirement all positions in the kingdom could be uniquely specified.

There were two schools of surveying in operation. One, the *daytime* school, used a compass to determine the direction of North. The other, the *nighttime* school, used the North star to determine the direction of North. As the sophistication of the measuring instruments increased, people began to notice that the daytime and nighttime measurements didn't quite agree. This is because magnetic North as determined by a compass is not in exactly the same direction as the North star. The figure to the right illustrates, although the actual difference is much less than in the diagram.



Finally, a young fellow named Albert attended both schools of surveying. He was also an irreligious person so did not take the religious requirement of measuring North-South distances in sacred units of feet seriously. To the consternation of the priests, he converted those North-South distances to everyday units by multiplying by k , the number of meters in a foot. He then discovered that although the daytime and nighttime numbers for the position of a particular place differed slightly, there was a combination of the measurements that was constant:

$$(k N_{\text{night}})^2 + (E_{\text{night}})^2 = (k N_{\text{day}})^2 + (E_{\text{day}})^2 \quad (\text{C.1})$$

What he is calculating, of course, is the distance squared between the town square and a particular location using Pythagoras' Theorem. You may wish to notice that there is no subscript on k : the conversion from feet to meters is the same for both schools of surveying.

The text shows that for two observers, say Sue and Lou, in relative motion who measure the time between two events and the distance between the same two events there is also a constant quantity. Here is Eqn. 37.19 of the text written in a more suggestive form:

$$(c \Delta t_{\text{Sue}})^2 - (\Delta x_{\text{Sue}})^2 = (c \Delta t_{\text{Lou}})^2 - (\Delta x_{\text{Lou}})^2 \quad (\text{C.2})$$

Notice that except for the minus sign, the form of this expression is very similar to the expression relating the daytime and nighttime measurements. One of the conclusions, then is:

The speed of light is only a conversion factor for units. If we had started out measuring time in everyday units of meters instead of sacred units of seconds, the speed of light would be exactly one. A time of 1 meter is the time it takes light to travel a distance of 1 meter.

Stop To Think C.1

Astronomers often measure lengths in units of *light years*. Say we measure time in units where the value is exactly one for the time it takes light to travel one light year. What is this unit for time?

C.2 Spacetime Diagrams

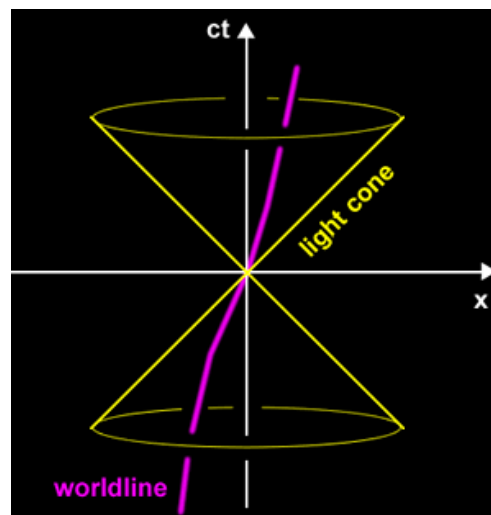
Here we discuss a useful way of visualising the nature of spacetime in Special Relativity: the *spacetime diagram*.

The time axis is vertical, and of course we have multiplied t by c so we are measuring time in meters, the same as the other coordinates.

The plot of time versus position for some object is called the *worldline*. Since we are not using "religious" units to measure time, the speed of light is exactly one. Therefore the worldline of an object moving at c makes an angle of 45 degrees, and is called the *light cone*.

A worldline for an object moving at less than the speed of light and that is at $x = 0$ at $t = 0$ is shown in purple.

The point $x = 0$ and $t = 0$ is called the *present*. The region of spacetime inside the light cone for positive values of the time is called the *future*. The region of spacetime inside the light cone for negative values of the time is called the *past*.



We can not know what happened at the star Alpha Centauri yesterday; it is about 4 light years away and since no information can travel faster than the speed of light we will have to wait four years to find out what happened there. Thus the coordinate of Alpha Centauri yesterday, which is outside the light cone, is inaccessible to us located in the present. Similarly, we can not send a signal to Alpha Centauri that will arrive tomorrow. Thus the entire region of spacetime outside the light cone is called *elsewhere*

Stop To Think C.2

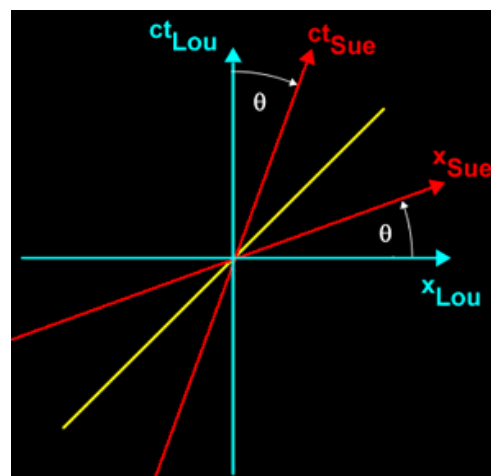
What is the orientation of a worldline for an object that is stationary? You may wish to specify your answer in terms of the angle it makes with the x axis.

C.3 The Significance of the Minus Sign

In our discussion of the text's *Spacetime Interval* subsection we saw that the equation relating coordinates in rotated reference frames, Eqn. C.1, has a very similar structure as the Relativistic one relating the time and distance between two events for two observers in relative motion, Eqn C.2., except that in Eqn C.2 there is an overall minus sign between the time and distance parts. The significance of the minus sign is that the space and time axes of the spacetime diagrams for the two observers, Sue and Lou, are rotated towards each other.

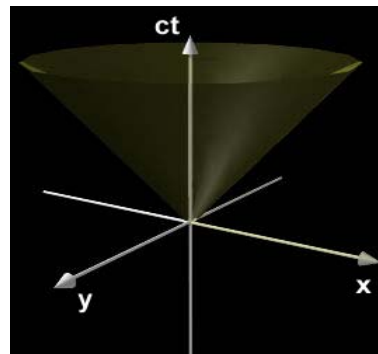
The angle between the axes is related to the relative speeds v of the observers by:

$$\tan(\theta) = \frac{v}{c} \quad (\text{C.3})$$



C.4 The Dimensions of Spacetime

There is a problem with the spacetime diagram: it only has one explicit spatial coordinate x . The way the light cone is drawn suggests, properly, that there is a second spatial coordinate, say y , that points out of the plane of the figure. Drawing a spacetime diagram with two spatial coordinates plus a time coordinate is not that difficult and an example is given to the right. But how do we indicate the third spatial coordinate, say z ?



The following figures indicate one way to approach a representation of such a four-dimensional object.

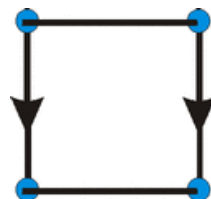
We begin with a zero-dimensional object, a point.



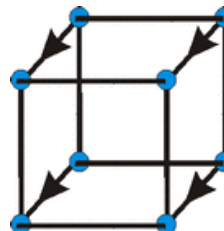
We move the point one unit to the right to generate a one-dimensional line.



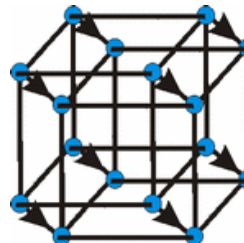
Moving the line one unit perpendicular to itself generates a two-dimensional square.



We move the square one unit perpendicular to itself, and we represent the three dimensional cube as shown.



Finally, if the moving of the square down and to the left was used to get from a square to a cube, then we represent moving the cube perpendicular to itself as moving it down and to the right. The result is called a *tesseract*.



In about 1884 Edwin Abbott wrote a lovely little book called *Flatland: a Romance of Many Dimensions*; the book has been reprinted many times and is readily available. In it he imagines a world with only two spatial dimensions. One of Flatland's inhabitants, named *A. Square*, became aware of the existence of a third spatial dimension through an interaction with a higher dimensional being, a *Sphere*. He attempted to explain this third dimension to the other inhabitants of Flatland, which of course promptly got him put in jail. The difficulties *A. Square* had in visualising the third spatial dimension is analogous to the difficulties we have in visualising a four-dimensional spacetime.

A very simple Flash animation of the interaction of the Sphere with Flatland is available via the blue button to the right. It will appear in a separate tab/window.



There is also a lovely video which explores the idea of Flatland, which may be accessed via the link to the right. Running time: 5:11.



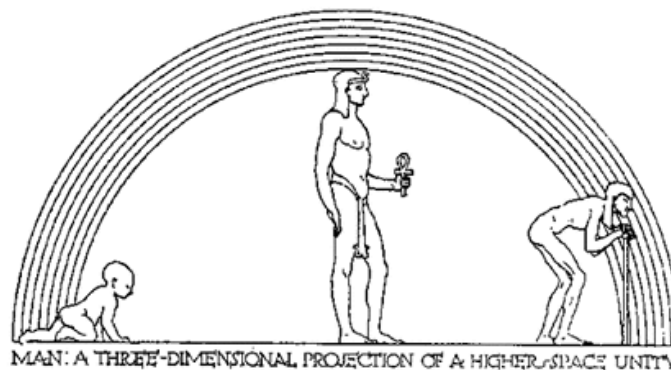
Here is another video about Flatland, which extends the discussion to tesseracts. It is by Carl Sagan. Running time: 7:15.



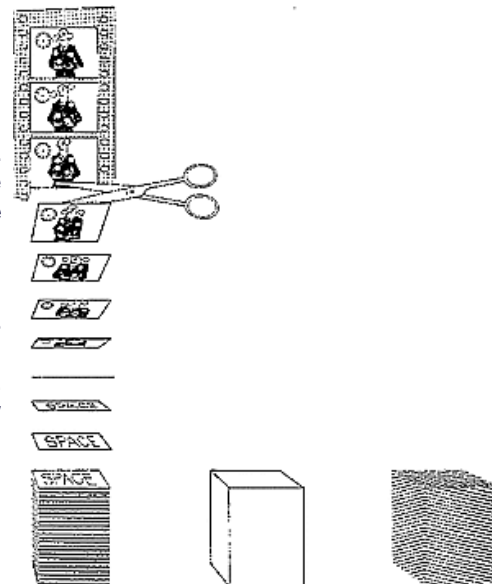
C.5 More Spacetime Diagrams and Some Discussion

The spacetime diagram to the right was drawn by Claude Bragdon in 1913 for his book *A Primer of Higher Space*. In this figure the time axis is horizontal.

Bragdon's "day job" was as an architect. He, along with Abbott, also believed that learning to comprehend a fourth dimension was in some sense equivalent to enlightenment. Bragdon designed many buildings in Rochester New York on which the tesseract can be found.



Here is another spacetime diagram, this time from D. Postle, *Fabric of the Universe*, pg. 106. It can be seen that different observers slice spacetime, the frames of the movie, in different ways.



We imagine our worldline in this spacetime diagram. Then, as David Park wrote, "our consciousness crawls along our worldline as a spark burns along a fuse" (in J.T. Fraser et al., eds., *The Study of Time*, pg. 113). As it crawls up our worldline we discover new slices of spacetime.

Louis de Broglie wrote a famous commentary on the worldview of the theory of relativity:

"In space-time, everything which for each of us constitutes the past, the present, and the future is given in block, and the entire collection of events, successive for us, which form the existence of a material particle is represented by a line, the world-line of the particle Each observer, as his time passes, discovers, so to speak, new slices of space-time which appear to him as successive aspects of the material world, though in reality this ensemble

of events constituting space-time exist prior to his knowledge of them." -- in Albert Einstein: Philosopher-Scientist , pg. 114.

Dogen Zenji seemed to have a similar view 800 years ago.

"It is believed by most that time passes; in actual fact it stays where it is. This idea of passing may be called time, but it is an incorrect idea, for since one only sees it as passing, one cannot understand that it stays just where it is. In a word, every being in the entire world is a separate time in one continuum." -- Shobogenzo.

Einstein wrote when his friend Besso died, "For us believing physicists, the distinction between past, present, and future is illusion, however persistent."

We conclude this part of our discussion on the meaning of spacetime with a famous Zen story. Two Zen monks were arguing about a flag waving in the breeze, and whether it was the flag or the wind that was moving. The Sixth Patriarch of Zen, Hui Neng, overheard; "I suggested it was neither, that what moved was their own mind."

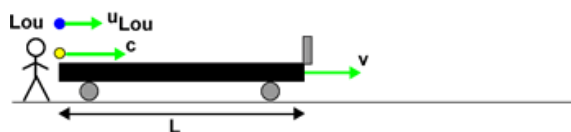
A commentary by Mumon may be appropriate:

Wind, flag, mind moves.
The same understanding.
When the mouth opens
All are wrong.

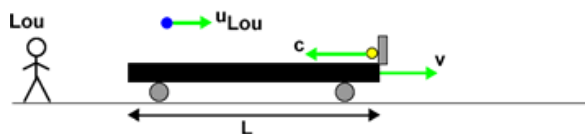
C.6 Addition of Velocities

In §37.8 the textbook shows that velocities for different observers do not transform as expected by our common sense. This section of the textbook and its main topic, the Lorentz transformation, are not part of the syllabus for this course. In this little section I derive the same result as the text, but use only the fact that the speed of light is the same value for all observers.

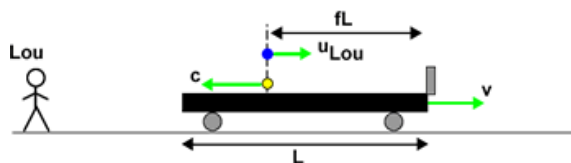
We imagine Lou watches a train go by with speed v . For him the length of the train is L . On the left side of the train a pulse of light travels towards the front of the train at speed c , and an everyday particle travels towards the front of the train at speed $u_{\text{Lou}} < c$. There is a mirror on the front of the train.



At a time t_1 the light is reflected from the mirror.



At a time t_2 later the light pulse meets the particle. They meet behind the front of the train by some fraction f of the full length of the train



The fraction f is some number between 0 and 1, and will have the same value for all observers. Perhaps there is an array of sensors on the floor of the train and one of the them lights up when the light and the particle meet up. All observers will agree on which sensor is lit, and therefore the value of f .

The total distance the particle moved from the start of the race to its encounter with the light is equal to the distance the light moved in going from the rear to the front minus the distance it moved in going from the front back to the particle.

$$u_{Lou}(t_1 + t_2) = c(t_1 - t_2) \tag{C.4}$$

The distance the light traveled in going from the rear to the front is just the length of the car plus the distance the train moved.

$$ct_1 = L + vt_1 \tag{C.5}$$

The distance the light traveled in going from the front to where it met the particle is the distance from the front of the train to the meeting point minus the distance the train moved.

$$ct_2 = fL - vt_2 \tag{C.6}$$

Eliminating L from Eqns C.5 and C.6 and combining with Eqn C.4 gives

$$f = \frac{(c+v)(c-u_{Lou})}{(c-v)(c+u_{Lou})} \tag{C.7}$$

Nowhere have we assumed that the speed of the train is non-zero. Therefore Eqn. C.7 is also valid for an observer, Sue, who is riding on the train.

$$f = \frac{(c-u_{Sue})}{(c+u_{Sue})} \tag{C.8}$$

Equating Eqns. C.7 and C.8 gives

$$u_{Sue} = \frac{u_{Lou} - v}{1 - \frac{u_{Lou}v}{c^2}} \tag{C.9}$$

D: General Relativity

This topic is not discussed in the textbook.

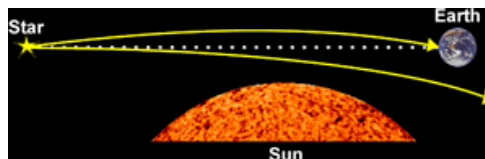
D.1 Three "Easy" Pieces

Einstein used three different ideas to build the General Theory of Relativity, which we describe here.

Piece 1 - Geometry is Physics

To the right are shown a distant star, the Sun, and Earth. Clearly the figure is not drawn to scale.

There is a straight dotted line connecting the star and the Earth. Imagine a light ray that leaves the star along the dotted line: it is headed directly for the Earth. Will that light ray actually reach the Earth?



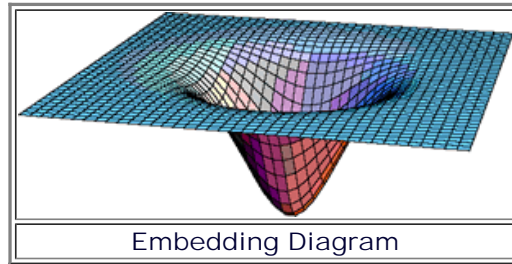
Let us think for a moment about the equation $E = mc^2$. We know that light has energy, and therefore it has mass. And *all* masses are attracted gravitationally by all other masses. So the light energy that left the star along the dotted line will be attracted gravitationally by the mass of the sun, which will cause its path to be deflected as shown. Thus that light ray from the star does not reach the Earth. It is the upper trajectory in the figure that represents the light from the Star that reaches the Earth.

But the definition of a straight line is that it is the shortest distance between two points. And we also know that light rays are the fastest way of getting information from, say, the star to the Earth. Thus, the upper path in the figure defines a straight line.

We imagine a flat plane as representing all three spatial dimensions plus the time dimension, a *Flatland* analogy.

Then, we are now led to the view that the effect of a mass such as the Sun is to curve spacetime, which we represent by an "embedding diagram."

We say that a flat spacetime has a *Euclidean geometry*. Then the effect of a mass is to make the geometry around itself non-Euclidean. And in a gravitational interaction everything is moving in "straight lines" in this curved spacetime. Technical note: sometimes one uses the word *geodesic* instead of saying "straight lines."



Beware: the surface is a representation of the three spatial dimensions plus the time dimension. Thus there is no physical dimension into which the surface can be distorted. Thus, the embedding diagram is just a visualisation technique.

"So long as one believes that the universe is a big machine, it is natural to think that its various parts can exert a force on one another. But the deeper science probes toward reality, the more clearly it appears that the universe is not like a machine at all. So Einstein's Law of Gravitation contains nothing about force. It describes the behavior of objects in a gravitational field - the planets, for example - not in terms of 'attraction' but simply in terms of the paths they follow." -- Lincoln Barnett, *The Universe and Dr. Einstein*, pg. 42

This view also seems consistent with a much earlier statement of Chinese philosophy:

"The inviolability of natural laws rest on [the] principle of movement along the lines of least resistance. These laws are not forces external to things, but represent the harmony of movement immanent in them." -- I Ching

It is also reminiscent of a graffiti: *Gravity is a myth; the Earth sucks!*

"Spacetime tells mass how to move; mass tells spacetime how to curve." -- Edwin F. Taylor and John Archibald Wheeler, *Exploring Black Holes*, back cover. This well known quote from Taylor and Wheeler also appears in many other places.

Piece 2 - Hiding the Universe in the Universe: The Principle of Equivalence sub-section of §13.3 of the text

We all know that we can describe the gravitational interaction with:

$$\vec{F} = -G \frac{mM}{r^2} \hat{r} \quad (\text{D.1})$$

We also know that we can describe the momentum of an object by:

$$\vec{p} = m\vec{v} \quad (\text{D.2})$$

Although both equations contain the mass m , a moment's reflection may convince you that they are describing very different properties of an object. In Eqn. D.1 we are talking about the *gravitational*

mass, the property that bodies have that cause them to interact gravitationally. In Eqn D.2 we are talking about the *inertial mass*, the property that causes bodies to have inertia. Why should those two very different properties of objects have the same value?

We can compare the gravitational mass of two objects by weighing them, i.e. measuring their gravitational interaction with the Earth. We can also compare the inertial mass of the same two objects by measuring their effects in collisions with other objects. These experiments have been done to great accuracy and precision, and the ratios always come out the same for gravitational and inertial mass. Thus we conclude that the gravitational and inertial mass are experimentally found to be the same for all objects.

In the 19th century Mach elevated the equality of gravitational and inertial mass into what is now called *Mach's Principle*. As we shall see, although we will be able to achieve some understanding of this principle, a complete and unambiguous statement of it is difficult, maybe even impossible.

One approach involves the *Foucault Pendulum*, which was invented by Jean Bernard Foucault (pronounced *foo-ko*) in 1851 in Paris and was demonstrated for the first time at the world's fair in the Pantheon in Paris.

The pendulum, the big brass ball in the figure, swings back and forth. However, the plane of the oscillation rotates as the earth turns under it. If the pendulum were at the North Pole, the plane of oscillation would make a complete revolution in 24 hours. From our perspective, it seems the entire universe circles us every 24 hours. However, the pendulum seems to show that it is the Earth that is rotating, not the universe



Foucault Pendulum at the University of Louisville

Newton's Laws tell us that every body will continue at rest or in uniform motion in a straight line unless forced to change that state of motion. And the Foucault pendulum is trying to obey that principle by rotating its plane of oscillation as the Earth turns under it. So this is a consequence of the inertial mass of the ball of the pendulum.

A Flash animation of the Foucault pendulum at the North Pole is available via the blue button to the right. It will appear in a separate tab/window.



Here is what Mach said about the Foucault pendulum:

"The universe is not *twice* given, with an earth at rest and an earth in motion; but only once, with its *relative* motions alone determinable. It is accordingly, not permitted us to say how things would be if the earth did not rotate." -- The Science of Mechanics, T.J. McCormack, trans., pg. 266.

What he is suggesting is essentially that the property of inertia of the Foucault pendulum *here* is related to its gravitational interaction with the mass of the universe *there*.

A Flash animation of the Foucault pendulum and Mach's Principle is available via the blue button to the right. It will appear in a separate tab/window.



Jay Orear stated the viewpoint of Mach's Principle nicely:

"The point of view taken here is that your head is the center of the universe and is always at rest. Whenever you bump your head against a wall, the distant galaxies have suddenly

accelerated and exert a strong gravitational force on your head. In order to keep your head 'at rest', the wall must exert an equal and opposite contact force. So the next time you bump your head or stub your toe, you can blame it on the distant galaxies." -- Fundamental Physics 2nd ed., pg. 288.

Incidentally, the title of this sub-section is from Chuang Tsu 6 .

Stop To Think D.1

Imagine a test mass m in a universe where there is no other mass. Will Conservation of Momentum be true for the mass?

Piece 3 - The Equivalence Principle

The realisation that Geometry is Physics is largely attributable to Riemann in the 19th century and Mach's Principle, also from the 19th century, is of course due to Mach. Here we describe Einstein's insight that allowed him to fuse Riemann's idea and Mach's Principle into the General Theory of Relativity.

Imagine that the room in which you are sitting is a cabin on a rocket ship. The cabin has no portholes so you can not look out. Devise an experiment to determine which of the following two situations are true:

1. The rocket ship is sitting stationary on the surface of the Earth, where the acceleration due to gravity is 9.8 meters per second per second down.
2. The rocket ship is in free space accelerating upward at 9.8 meters per second per second.

It turns out that for a reasonably small cabin on the rocket ship, no such experiment can be devised. In 1908 Einstein elevated this to a general principle:

"We shall therefore assume complete physical equivalence between the gravitational field and the corresponding acceleration of the reference system. (Reference: Jahrbuch der Radioaktivitat un Elektronik 4 (1908) pg. 443.)

We call this *Einstein's Equivalence Principle*, which he later described as "the happiest thought of my life."

Einstein once stated the principle as:

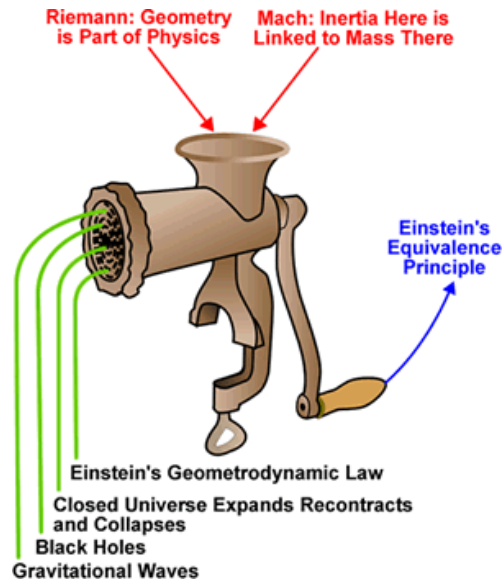
"For an observer in free fall off the roof of his house, there exists for him during his fall no gravity."

At the very beginning of this document I stated that building a theory that considers observers in any state of relative motion, including relative acceleration, would also turn out to be a theory of gravitation. The Equivalence Principle is why this is true. A few years ago a student wrote a paper which had an appropriate typo: *gravitivity*.

If you think about this sub-section and the previous one, you may come to the conclusion that the equivalence of inertial and gravitational mass is intimately related to the equivalence of acceleration and gravitation. This conclusion is correct, which is why Knight calls the fact that inertial and gravitational mass are equal the "Principle of Equivalence." In these notes I make a small distinction between these two ideas.

D.2 Einstein's Synthesis

The following figure neatly summarises Einstein's achievement. It is based on a figure by J.A. Wheeler, "From Relativity to Mutability" in J. Mehra, ed., The Physicist's Conception of Nature (Reidel, 1973), pg. 205.



You may wish to know that the General Theory, built from three elegant and beautiful pieces, is mathematically very complex. However, the worldview that it gives us of the nature of space and time is also elegant and beautiful.

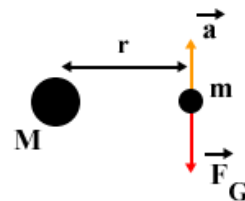
Some chronology may be interesting here. Einstein published the Special Theory of Relativity in 1905, based on 10 years of struggle with the question of what he would see if he pursued a beam of light at the speed of light. He thought of the Equivalence Principle a couple of years later, publishing it in 1908. He did not complete and publish the General Theory of Relativity until 1915, although he had been working hard on the problem since 1908. As he wrote to Sommerfeld in 1912:

"I am exclusively occupied with the problem of gravitation and hope with the help of a local mathematician friend [Marcel Grossman] to overcome all the difficulties. One thing is certain, however, that never in life have I been quite so tormented. A great respect for mathematics has been instilled within me, the subtler aspects of which, in my stupidity, I regarded until now as pure luxury."

More About Mach's Principle

According to General Relativity, in addition to the usual gravitational field proportional to $1/r^2$, for accelerating masses there is an additional field proportional to $1/r$.

Although General Relativity does not talk about forces, we can approximate the curvature of spacetime as a force. If two masses m and M have a relative acceleration a , then the $1/r$ field produces a force in the opposite direction to the relative acceleration that is approximately equal to:



$$F_G \approx G \frac{Mma}{c^2 r} \tag{D.3}$$

If we sit on a mass m that is accelerating relative to the galaxies, the total force exerted on m is the sum of Eqn D.3 for all the galaxies.

$$F_G \approx \sum_i G \left(\frac{ma}{c^2} \right) \left(\frac{M_i}{R_i} \right) \tag{D.4}$$

Assuming an average density of the matter in the universe is ρ and a maximum radius R , then integrating Eqn D.4 gives:

$$F_G \approx \frac{G\rho R^2}{c^2} ma \quad (\text{D.5})$$

As discussed below, the radius of the visible part of the universe is about 15 billion light years. As also discussed below, estimating the average density of the universe is difficult, but perhaps a reasonable value is 10^{-25} kg/m³. Then:

$$\frac{G\rho R^2}{c^2} \approx 0.9 \quad (\text{D.6})$$

Thus within experimental uncertainties we have derived Newton's Second Law from General Relativity.

Stop to Think D.2

Do you think an accelerating charge would produce an electric field proportional to $1/r$ in addition to the usual inverse square field?

D.3 Predictions of General Relativity

Here we briefly describe some of the predictions of the General Theory of Relativity and their experimental tests.

Gravitational Time Dilation

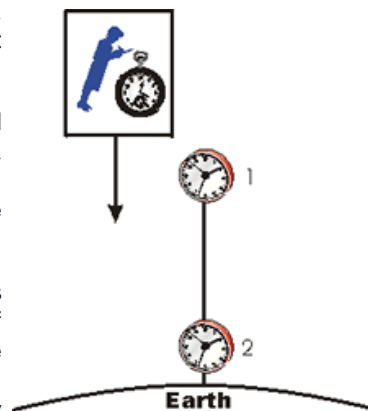
In Special Relativity moving clocks run slow. Similarly, clocks in gravitational fields run slowly. This prediction of the theory has been tested and confirmed many times.

Deriving Gravitational Time Dilation

Here we qualitatively derive gravitational time dilation. The "derivation" uses Einstein's Principle of Equivalence and the idea we first encountered in Special Relativity: the time for some event is measured by a clock right beside where the event occurs.

We imagine we have two clocks, labeled 1 and 2, that are fixed and stationary relative to the surface of the Earth. We are in a reference frame that is in free fall towards the surface of the earth, and we have our own clock, stationary with respect to us. Note that since we are in free fall, we are floating.

According to the Equivalence Principle, our reference frame is inertial, and therefore our clock can do good measurements of time. Note that any other clock, stationary relative to us, at some other location is not necessarily doing good measurements of time. Thus we will compare the rates of the Earth clocks to ours only when we pass right by them.



When we pass by Clock 1, it is moving with respect to us. Therefore, Special Relativity tells us that it is running slowly relative to our clock. Similarly when we pass by Clock 2, since it is moving with respect to us it will be running slowly compared to our clock.

But, since we are in free fall our speed with respect to the Earth and the two Earth-bound clocks is

increasing: we are accelerating down at 9.8 m/s^2 relative to the Earth. So when we pass Clock 2, its speed with respect to us is greater than the speed of Clock 1 when we passed by it.

Thus, for us in our good inertial reference frame we conclude that Clock 2 is running more slowly than Clock 1. So the clock in the stronger gravitational field, Clock 2, runs more slowly than the clock in the weaker gravitational field

Global Positioning Systems

This effect is crucial in Global Positioning Systems. As you may know, there are over two dozen GPS satellites in orbit around the Earth. Each satellite has an extremely accurate atomic clock, and broadcasts the time as measured by its own clock. The GPS receiver receives the signals from the satellites, and by measuring the time delay between when the signal was sent and when it was received calculates where it is. However, if corrections are not made for the facts that the clocks in the satellites are moving relative to the Earth and are in weaker gravitational fields, the position measured by the receiver can drift by up to 11 km for every day that the satellite has been in orbit.

Stellar Aberration

We began the discussion of General Relativity by discussing the fact that a massive object will curve the trajectory of light passing near it. Thus the apparent position of a distant Star will be different from its real position if the light from it passes near the Sun. This prediction of the theory has been tested and confirmed many times.

One of the first such confirmations, by Eddington in 1919, and Einstein's attitudes towards his theory is provided in the following account by Ilse Rosenthal-Schneider:

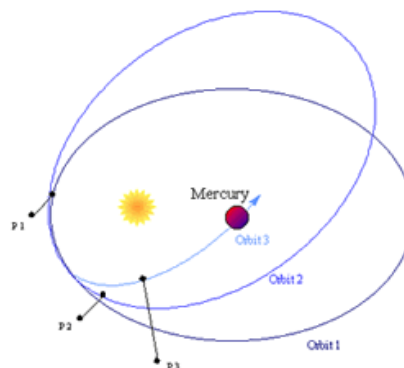
"Einstein was discussing some problems with me in his study when he suddenly interrupted his explanation and handed me a cable from the windowsill with the words, 'This may interest you.' It was the news from Eddington confirming the deviation of light rays near the sun that had been observed during the eclipse. I exclaimed enthusiastically, 'How wonderful, this is almost what you calculated.' He was quite unperturbed. 'I knew that the theory was correct. Did you doubt it?' When I said, 'Of course not, but what would you have said if there had not been such a confirmation?' He retorted, 'Then I would have to be sorry for dear God. The theory is correct.'" -- H. Woolf, ed., *Some Strangeness in the Proportion* (Addison-Wesley, 1980), pg. 523.

Advance of the Perihelion

According to classical Newtonian mechanics, the planets move in elliptical orbits around the Sun. These orbits, in the absence of perturbations from other planets, comets, etc., should exactly repeat themselves.

In General Relativity these ellipses do not exactly repeat. The point of closest approach of a planet to the sun is called the *perihelion*, which has the same root as *periscope*.

The figure shows this "advance of the perihelion" predicted by General Relativity, although the advance is not nearly as large as the figure indicates.



Figure

source:

http://astrosun.tn.cornell.edu/courses/astro201/merc_adv.htm

This prediction of General Relativity has been experimentally tested for the orbit of Mercury. The measurements are difficult but it is probably fair to say the prediction is confirmed.

A Flash animation comparing Newton and Einstein's prediction of the orbit of Mercury may be accessed via the blue button to the right. It will appear in a separate tab/window.



Gravity Waves

Just as an oscillating electric charge generates an electromagnetic wave, General Relativity predicts that an oscillating mass will generate a gravity wave. Many attempts have been made to detect the existence of gravity waves, with results that are ambiguous at best.

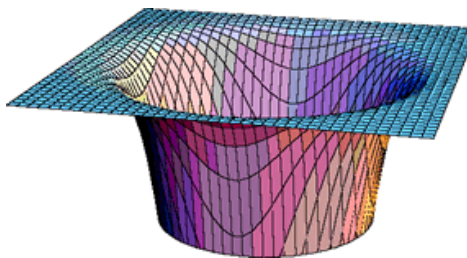
There are currently some experiments that are being designed to attempt to detect these waves. One large project is the Laser Interferometer Gravitational-Wave Observatory (LIGO). The home page of the project is <http://www.ligo.caltech.edu/>. You may wish to note that the experiment uses huge Michelson Interferometers.

Black Holes

When a gravitational field, a curvature in spacetime, becomes sufficiently intense, the theory predicts that the matter causing the field can be literally crushed out of existence, although the curvature of spacetime remains. The curvature is so intense that a hole is punched in the fabric of spacetime. This phenomenon is called a *black hole*.

The reason why these objects are called black holes is because at some distance away from the central *singularity* the curvature of spacetime becomes so intense that not even light can escape.

There are many observations that strongly suggest the existence of black holes that were formed when a star exhausted its nuclear fuel.



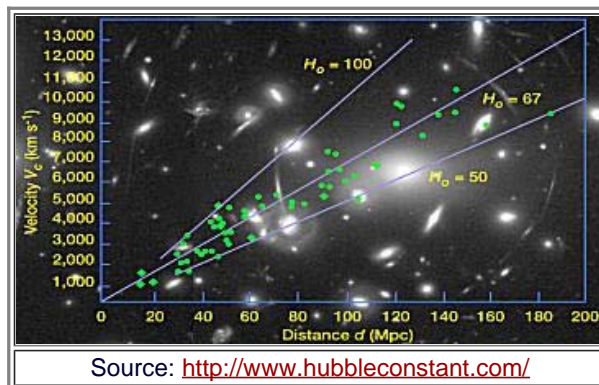
Expanding Universe

In 1915, when Einstein was finishing up his General Theory he realised that it predicted an expanding universe. However, the permanence of the universe was a fixed item of Western philosophy. This conflict between a dynamic non-permanent universe and the firm philosophical beliefs of the day bothered Einstein greatly. In fact, he weakened and modified his theory, introducing a *cosmological constant*, a fudge factor to make the universe static.

In 1929 Edwin Hubble was doing some interesting observational astronomy. It was known that there is a class of stars called *Cepheid variables* whose intensity varies periodically. Further there was a known relationship between the absolute brightness of a Cepheid variable star and the rate of the oscillation of its intensity. Thus by measuring the brightness of such a star on Earth we can calculate how far away the star is.

In addition, the spectrum of the light from a star when the star is at rest relative to us was also known. However, if the star is, say, moving away from us the Doppler effect will cause the wavelengths of the light to be shifted towards the red. You have probably noticed the Doppler effect when a car playing loud music goes by you: as it approaches the sound is shifted towards shorter wavelengths and as it recedes the sound is shifted towards longer wavelengths. So by measuring the shift in the spectrum of light from a Cepheid variable we can determine its speed relative to us.

When Hubble did these measurements he discovered that the stars are moving away from us, and the further away they are the faster they are receding from us. The figure to the right shows a more modern set of data.



Here is a model that explains how such data can arise. We imagine that we are making raisin bread, but have misread the recipe and instead of putting in one teaspoon of baking powder we have put in one pound! As the dough starts to rise, it begins to fill our kitchen. If we stand on any given raisin, all the other raisins will be receding away from us; the further away the raisin is the faster it recedes away from us. Similarly, if the universe is expanding and the raisins represent the stars we will get a result similar to the one observed by Hubble.

Here is a more accurate model. We imagine the universe as a Flatland confined to the surface of a balloon. Paint some dots on the balloon and think of them as the stars. Begin blowing up the balloon. If you stand on any dot and measure the position and speed of any other dot you will find that all the dots are receding away from you and their speed away from you increases the further away the dot is. Note that the dots are not moving on the surface of the balloon, rather the fabric of the balloon itself is expanding. Also remember that you must make your distance measurements staying within the surface of the balloon; the higher-dimensional direct connection from point to point via a *chord* inside the balloon is not accessible to us.

When Hubble published his results, Einstein immediately dropped the cosmological constant, calling it "the biggest mistake of my life."

Note that the Hubble data, as well as the two models we have used to think about it, indicate that at some time in the past the universe was very very small. In fact, current cosmology tends to favor a view that at some time on the order of 15 billion years ago the size of the universe was **zero**. At this time, the temperature was infinite, and the *Big Bang* occurred, initiating this cycle of the universe.

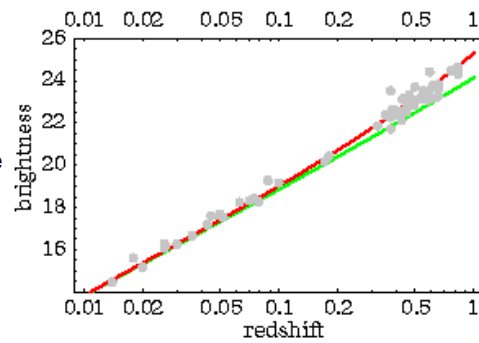
D.4 Problems

Until recently Big Bang cosmology was thought to be well understood. It accounted for a great deal of data regarding the large scale structure of the universe. Some problems were known, including:

- The flatness problem : Why is the matter density of the universe so close to the unstable critical value between perpetual expansion and re-collapse into a Big Crunch?
- The horizon problem : Why does the universe look the same in all directions when it arises out of causally disconnected regions? This problem is most acute for the very smooth cosmic microwave background radiation.
- The dark matter problem : Of what stuff is the Universe predominantly made? Analysis of the gravitational interactions of galaxies shows much more matter than we can see. Nucleosynthesis calculations suggest that this *dark matter* of the Universe does not consist of ordinary matter - neutrons and protons?

Then in 1998 Perlmutter et al. published data that showed that, contrary to expectations, the rate of expansion of the universe is actually *increasing*. They measured the brightness and redshift of supernovae. The brightness is a direct measure of their distance away from us, and the redshift measures the speed of the supernovae away from us. Thus Perlmutter was taking the same sort of data as Hubble did 70 years before, but this time for supernova, which are much further away from us than the Cepheid variable stars that Hubble used.

For the data to the right the upper red line is for a finite universe whose rate of expansion is increasing.



This experiment was recently replicated by NASA, and the Perlmutter's data was duplicated with better precision and accuracy.




This makes no sense. We expect that all the mass of the universe is gravitationally attracting all the other mass. Thus we expect the rate of expansion of the universe to be decreasing, not increasing. Various attempts are currently underway to explain this mysterious result. Two of those attempts are:

- Dark Energy. This is some sort of unobservable substance causing the rate of expansion to increase. Nobody has any idea what such energy is. It is almost some sort of anti-gravity pushing all the masses of the universe away from all the other masses.
- Cosmological Constant. Just as Einstein introduced a fudge factor into General Relativity to force the universe to match his pre-conceived idea of what the universe is like, this is a new fudge factor introduced to force the rate of expansion of the universe to increase.

This may be a very exciting time for Physics: it is possible that the General Theory of Relativity, time tested for over 90 years, is collapsing!

Other Animations

Here are some other animations illustrating the Theories of Relativity. They may all be accessed via the blue button, and will open in a new tab/window.

A Flash animation on Time Dilation in Special Relativity	
A Flash animation on Length Contraction in Special Relativity	
A Flash animation on Simultaneity in Special Relativity	

Note: In the Length Contraction animation, I state that the lifetime of a muon at rest relative to us is 2.196 us. This is the *mean* lifetime. The text on page 1159 also discusses muon decay and states that the *half-life* is 1.5 us, which is also correct. The mean lifetime is not the same as the half-life. The difference is not important for our purposes.

Stop To Think Answers

Stop To Think C.1

Years. Light travels a distance of one light year in a time of one year. In this system of units, the speed of light is exactly 1 light year per year.

Stop To Think C.2

Vertical. It makes an angle of 90° with the x axis. The x coordinate of the object does not change.

Stop To Think D.1

No. Conservation of Momentum states that the inertial mass m_{inertial} times the speed v is constant. But according to Mach's Principle the inertial mass is zero.

Note that in a universe with only this test mass, we and our apparatus to measure its momentum could not exist.

Stop To Think D.2

Yes. This is the field that causes a wave of electromagnetic radiation.

Suggested Exercises and Problems

SCN 1

Assume that the speed of light is c relative to the ether. The ether wind v is blowing horizontally towards reflector R_2 at 3.00×10^4 m/s. The lengths of the arms of the interferometer are equal, with value $L = 10.0$ m. Light of frequency $f = 6.00 \times 10^{14}$ Hz is emitted by the source.

- Verify Equation A.1.
- When the light reaches the detector, what is the difference in phase of the two beams?
- If the interferometer is rotated by 90 degrees, what is the difference in phase of the two beams?
- Would the difference between your answers to Parts B and C be observable?

SCN 2

Show that the relativistic Doppler effect, Equation 20.41, is correct.

SCN 3

Relative to some observer Event 1 occurs at position x_1 and time t_1 . Event 2 occurs at position x_2 and time t_2 where t_2 is $> t_1$. Imagine that some signal is generated by Event 1 which arrives at the position of Event 2 just as it occurs.

The interval squared s^2 defined by Equation 37.19 of the text can be negative, zero, or positive. For each of these three cases, what must be the speed of the signal going from Event 1 to Event 2?

SCN 4

- Show that Eqn C.7 gives a reasonable result for $u_{\text{Lou}} = v$, i.e. the particle is not moving relative to the train.
- Show that Eqn C.7 gives a reasonable result for $u_{\text{Lou}} = c$, i.e. the particle is moving at the same speed as the light.
- (Challenging) Assume that the speed of light is c relative to the ether, and that Lou is stationary relative to the ether. Show that the equivalent of Eqn C.9 is the common sense expression for the addition of velocities.

SCN 5

Special Relativity tells us that spacetime has four dimensions, the three spatial ones and time. Thus the relativistic equivalent of the position vector is:

$$\vec{s} = (cs_t, s_x, s_y, s_z)$$

The relativistic equivalent of the magnitude squared of the position vector is the interval squared

$$s^2 = (cs_t)^2 - s_x^2 - s_y^2 - s_z^2$$

The value of the interval squared is *invariant* with the same value in *all* inertial reference frames.

The textbook derives another invariant quantity, Equation 37.45, which can be written in terms of a four dimensional vector momentum with magnitude squared given by

$$p^2 = E^2 - cp_x^2 - cp_y^2 - cp_z^2$$

What is the time component of the four dimensional vector momentum?

Solutions

Reminder: looking at solutions to problems you have not made a good try at solving is usually not of benefit. You may see the solutions to the problems by clicking [here](#).

Author & Sources

These Notes were written by David M. Harrison, Dept. of Physics, Univ. of Toronto, in February 2009.

In March 2009 PHY132S student Dylan Jacobs contributed to the discussion of GPS systems, PHY132S student Bernard Kadosh bought the Carl Sagan video about Flatland and tesseracts to my attention, and PHY132S student Michael Berg contributed to the discussion of the Foucault pendulum. In April 2009 PHY132S Learning Assistant (aka "Teaching Assistant") Omar Gamel contributed to the discussion of the significance of the minus sign in the expression for the interval.

Much of the material on Special Relativity is contained in a larger non-mathematical document I wrote in 1999:

<http://www.upscale.utoronto.ca/PVB/Harrison/SpecRel/SpecRel.html>.

Much of the material on General Relativity is contained in a larger non-mathematical document I wrote in 2000:

<http://www.upscale.utoronto.ca/PVB/Harrison/GenRel/GenRel.html>.

The Parable of the Surveyors is from Edwin F. Taylor and John Archibald Wheeler, Spacetime Physics (Freeman, 1966), pg. 1. This classic book is highly recommended.

Section C.6 is based on N. David Mermin, "Relativistic addition of velocities directly from the constancy of the speed of light," American Journal of Physics 51(12), December 1983, pg. 1130 - 1131.

The **More About Mach's Principle** sub-section is based on Jay Orear, Fundamental Physics (John Wiley & Sons, 1967), pg. 287 - 288.

Last revision: \$Date: 2010/08/21 10:54:26 \$ (y/m/d UTC).