\Lecture 3

Plane Waves, Phasors, Refractive Index and Dispersion

Maxwell's Equations

- Div[ϵ_0 **E**] = - Div[**P**] + ρ
- Div[μ_0 **H**] = - Div[μ_0 **M**]
- Curl[**E**] = - $\partial/\partial t$ (μ_0 **H**) - $\partial/\partial t$ (μ_0 **M**)
- Curl[**H**] = $\partial/\partial t$ (ϵ_0 **E**) + $\partial/\partial t$ (**P**) + **J**
- $\mu_0 = 4\pi \cdot 10^{-7}$ in SI units
- $\epsilon_0 = 10^{-9}/(36\pi)$ in SI units

Wave Equation

- For a **linear**, **isotropic**, **charge-free** medium
- Del2[**H**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**H**) - $\partial/\partial t$ ($\mu\sigma$**H**) = 0
- Del2[**E**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**E**) - $\partial/\partial t$ ($\mu\sigma$**E**) = 0
- The third term in the above equation represents the dissipation
- For a **linear**, **isotropic**, **charge-free**, **non-conducting** medium ($\sigma = 0$) the wave does not dissipate
- A Vacuum is a particular case of the above!!

Transverse Waves

- Del2[**H**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**H**) = 0
- Del2[**E**]] - $\partial^2/\partial t^2$ ($\mu\epsilon$**E**) = 0
- The most general plane wave solutions of the above are
- **E** = **x** [f_+ (z - vt) + f_- (z + vt)]
- **H** = (ϵ/μ)$^{1/2}$ **y** [f_+ (z - vt) + f_- (z+vt)]
- Note that **E** and **H** are perpendicular to each other and to the direction of propagation - z

Monochromatic Waves

- **E** = E_0 **x** cos(**k** . **r** ± ωt + ϕ)
- ϕ is an arbitrary phase
- ω is the angular frequency
- **k** defines both the direction of propagation (DOP) and the wave vector |**k**| = 2π/λ = k
- **k** . **r** ± ωt defines a surface of constant phase which is displaced in time at a speed ω/k which is called the "phase speed"

Velocity In a Medium

- If χ_e, χ_m are # 0, then the phase speed is reduced
- Phase speed becomes c/n
- n = $(\epsilon/\epsilon_0 \, . \, \mu/\mu_0)^{1/2}$
- n is the "refractive index" and is > 1 (mostly)
- If either of χ_e, χ_m are dependent upon frequency, then the medium has "dispersion"
- This means that if n is frequency-dependent, then the medium is dispersive

Complex Numbers

- Complex numbers for physicists are a "convenience" for solving problems
- Rely on the real-world fiction that √-1 can be represented by i (the engineers use j)
- A complex number can be written as C = a + ib
- The conjugate complex is C* = a - ib
- The modulus is |C| = $\sqrt{(a^2 + b^2)}$
- The argument is Arg(C) = $\tan^{-1}$ (b/a)

Complex Exponentials - Phasors

- If C = cos(a) + i sin(a)
- Then it is shown in all the best text books that you can write C = exp(ia)
- In fact any complex number C can be written as C = |C| exp(i arg(C))
- Since manipulation of exponentials in integration and differentiation is very easy, it is tempting to try to manipulate any wave equation into this format

Complex Superposition

- If E_0 x cos (k . r - ωt + ϕ) is a solution
- Then so is E_0 x sin (k . r - ωt + ϕ)
- And so is i . E_0 x sin (k . r - ωt + ϕ)
- By the principle of linear superposition, the sum of the above solutions is also a solution which can be expressed as
- E_0 x exp i(k . r - ωt + ϕ)
- or x . *E* . exp(-iωt) where *E* is complex and contains all spatial and phase components

Differentiation wrt Time

- For any quantity C exp(-iωt) where C is independent of time
- ∂/∂t (C exp(-iωt)) = -iω C exp(-iωt)
- Similarly Int[C exp(-iωt)] = -1/iω C exp(-iωt)
- What does this do to Maxwell's equations?
- Div[ϵ_0 ***E***] = - Div[***P***] + ρ - unchanged
- Div[μ_0 ***H***] = - Div[μ_0 ***M***] - unchanged
- Curl[**E**] = - ∂/∂t (μ_0 **H**) - ∂/∂t (μ_0 **M**) becomes
- Curl[***E***] = iωμ_0 ***H*** + iωμ_0 ***M***
- Curl[**H**] = ∂/∂t (ϵ_0 **E**) + ∂/∂t (**P**) + **J** becomes
- Curl[***H***] = - iωϵ_0 ***E*** - iω ***P*** + ***J***

ME Monochromatic Wave Eqn

- Del2[***H***]] + ω^2μϵ***H*** + iωμσ***H*** = 0
- Del2[***E***]] + ω^2μϵ***E*** + iωμσ***E*** = 0
- $\mu_0 = 4\pi \, . \, 10^{-7}$ in SI units
- $\epsilon_0 = 10^{-9}/(36\pi)$ in SI units
- If we now REDEFINE ϵ as being ϵ + iσ/ω, we can automatically incorporate conducting media in the solutions with a complex dielectric constant
- This implies that the effect of conductance is the the polarisation **P** is still proportional to the field **E**, but there is a phase lag between the field and the polarisation
- We can also look at the same arguments for the spatial differential Del2[] and find that in a charge-free medium Del2[***X***] = -k^2 ***X*** where k is the wave vector

Dispersion Relation

- Putting all these components into the wave equation, we find that
- $k^2 = \omega^2$ ϵ μ for a linear, isotropic, charge-free medium
- Remember that ϵ is generally complex and therefore so is k
- If we define refractive index to be complex following ϵ, then
- $k^2 = \omega^2 / c^2 \, . \, n^2$
- This is the "dispersion relationship" and measures how the wave vector k, varies with frequency

n and ϵ

- Most materials are non-magnetic, assume $\mu = \mu_0$
- $\epsilon = \epsilon_r + i\, \epsilon_i = n^2 \epsilon_0$
- So if $n = n_r + i\, n_i$
- Then $\epsilon_r / \epsilon_0 = n_r^2 - n_i^2$
- And $\epsilon_i / \epsilon_0 = 2\, n_r\, n_i$
- What's the point?
- We can measure the refractive index very easily
- therefore we can relate the macroscopic n to the microscopic, complex ϵ

Bringing It Back to Reality

- The wave propagates like exp i(**k** . **r** - ωt + ϕ)
- Suppose it propagates along the z axis -> kz
- The wavelength in free space is λ_v
- and the wavevector is k_v
- Look at the spatial part only
- $\exp(ikz) = \exp\{i\, (n_r + i\, n_i)\, k_v\, z \}$
- $\exp(ikz) = \exp(i\, n_r\, k_v\, z)\ \ \exp(-\, n_i\, k_v\, z)$
- First term is phase factor, second is a decay term
- Measure the phase speed - gives n_r (historically n)
- Measure the decay - gives n_i (historically κ)

Refractive Indexes

- Vacuum = 1.00 (surprise, surprise)
- Glasses = 1.57-1.77
- Sapphire = 1.77, Diamond = 2.42
- Silicon = 3.8 + 0.4i
- Silver = 2.3 + 3.8i
- Air = $1.0 + 290\ .\ 10^{-6}\ (\rho/\rho_{stp})$