Lecture 6

Group Velocity, Poynting and Polarisation

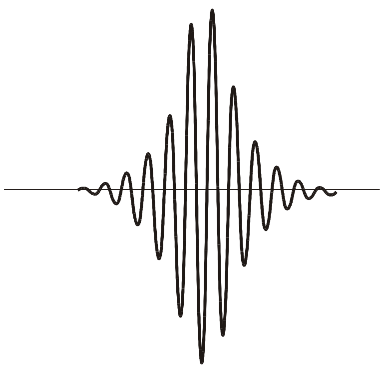
Group Velocity

- Phase velocity is the rate of advance of constant phase surfaces.
- Phase velocity can exceed velocity of light in vacuum, c.
- BUT phase velocity really only applies to infinite (time and space) monochromatic (sine) waves
- These do not exist - or if they do, are boring!!

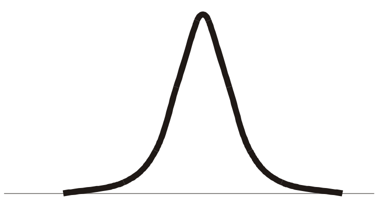
Propagation of Pulses

- Consider a gaussian pulse in time of width Δt - ie a sine wave multiplied by a gaussian
- By Fourier transforms we can show that this can be resolved into a superposition of sine waves, width $2/\Delta t$
- In frequency space this is a delta function convolved with a Gaussian

Time Space



Frequency Space



Representation

- Any superposition of sine waves can be represented as
- $E(t,z)$ = Int[$E_0(\omega)$ exp i(k(ω) z - ωt), dω]
- If the integrand is non-zero about a range of frequencies around ω_0 only
- Change variable to $\Delta\omega = \omega - \omega_0$
- $k(\omega) = k(\omega_0) + dk/d\omega|_{\omega 0}\, \Delta\omega$ (Taylor expansion)
- $E(t,z)$ = exp i(k(ω_0) z - ω_0t)
  Int[$E_0(\omega)$ exp -iΔω (t - $dk/d\omega|_{\omega 0}$ z) dω]
- First term is a carrier wave, propagating at the phase speed
- Second term is the modulation (information) traveling at the group velocity $d\omega/dk|_{\omega 0}$
- $d\omega/dk = c / n(\omega) / (1 + (\omega/n)\, dn/d\omega)$
- The group velocity should be smaller than the speed of light....

Group Velocity in Plasmas

- $n^2 = 1 - \omega_p^2 / \omega^2$
- If we assume that $\omega_p < \omega$, then n is real and < 1
- The phase velocity is c/n which is > c
- It is left as an exercise for the reader to show that....
- The group velocity is cn which is < c
- This is not a general formula - it only applies to plasmas
- The group velocity is the velocity at which information can propagate
- There is no information in a monochromatic plane wave

MEs in Linear, Isotropic Medium

- Div[**E**] = ρ/ϵ
- Div[**H**] = 0
- Curl[**E**] = - $\partial/\partial t$ (μ **H**)
- Curl[**H**] = $\partial/\partial t$ (ϵ **E**) + **J**
- $\epsilon = \epsilon_0 (1 + \chi_e)$
- $\mu = \mu_0 (1 + \chi_m)$
- Notice that **J** has not been replaced with the conductivity

Poynting Vectors

- Start with Div[**E** x **H**] = **H** . Curl[**E**] - **E** . Curl[**H**]
  (general vector relationship)
- Div[**E** x **H**] = - **H** . $\partial/\partial t$ (μ **H**) - **E** . $\partial/\partial t$ (ϵ **E**) - **E** . **J**
- Div[**S**] + $\partial/\partial t$ ($\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2$) + **E** . **J** = 0
- Where **S** = **E** x **H**
- The last term is a Joule heating term
- The second term is the rate of change of energy in the magnetic and electrostatic fields
- The first term can be interpreted as the energy flow with the Poynting vector pointing in the direction of energy flow
- Since **S** = **E** x **H** these form a right-handed set as expected from plane wave consideration

Poynting and Plane Waves

- **S** = **E** x **H**
- Let's take time averages
- <**S**> = < Re(**E**) x Re(**H**)> = ½ Re (**E** x **H***)
- But **H*** = $\sqrt{(\epsilon/\mu)}$ **k** x **E***
- So <**S**> = ½ $\sqrt{(\epsilon/\mu)}$ $|E_0|^2$ **k**
- The Poynting vector points in the direction of energy flow - the DOP in this case
- The magnitude is proportional to $|E_0|^2$

The General (Ugly) Case

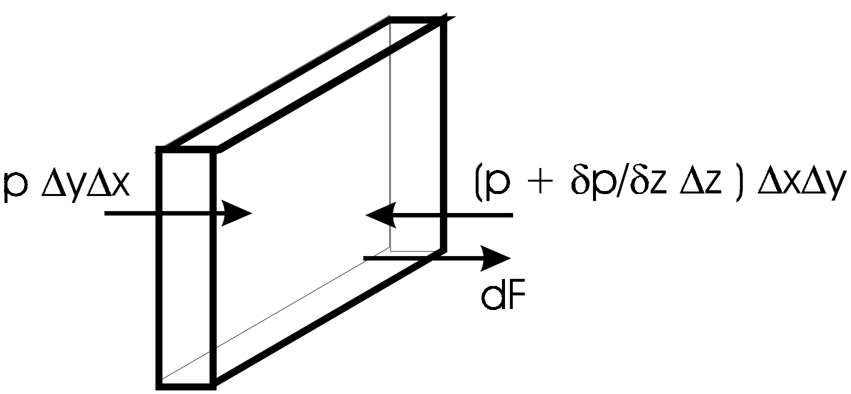
- Div[***E*** x ***H****] - iω(μ ***H***.***H**** + ϵ* ***E***.***E****) + ***E***.***J**** = 0
- Real part of this equation as the power equation
- Power dissipated if the real part of 2nd term <> 0
- Only true if μ, ϵ are complex (imaginary <> 0)
- Joule Heating occurs through the real part of ***E***.***J****
- If **J** = σ**E**, then the real part of σ counts
- Lossless medium has real μ, ϵ and imaginary or zero σ

The Force of a Plane Wave

- A plane EM wave passes through a conducting region
- A current **J** appears because of the electric field acting on the electrons
- A force **F** appears because of the interaction of the moving electrons with the magnetic field
- **F** = **J** x μ**H** (per unit volume)
- But we can substitute from Mes
- **F** = (Curl[**H**] - ∂/∂t (ϵ**E**)) x μ**H**
- After some substitution (it is left as an exercise..)
- **F** = -μ**H** x Curl[**H**] - ϵ**E** x Curl[**E**] - ϵμ∂/∂t (**E**x**H**)
- Third term is zero in steady state
- In plane wave only E_x, H_y <> 0 and ∂/∂x, ∂/∂y=0
- **H** x Curl[**H**] -> ½ ∂/∂z (H_y^2) **k**
- $F_z = -½\ \partial/\partial z\ (\mu H_y^2 + \epsilon E_x^2) = -\partial U/\partial z$
- Force per unit volume = rate of decay of energy in waves with distance

The Pressure of the Wave

- Force per unit volume is also the rate of change of pressure with distance (eg hydrostatic equation)
- Therefore the pressure p is equal to the energy density U
- $U = ½\ (\mu H_y^2 + \epsilon E_x^2)$ for the plane wave



The Pressure of the Wave

- Pressure p = |**S** |/c
- In the general case this needs integrating over direction to account for solid angles - but we are ignoring that and using plane waves
- Pressure is force/unit area
- Force is rate of change of momentum p
- ∂p/∂t = A P = A(1/c) |**S**|
- For a plane wave momentum/unit area is the energy in the beam/unit area divided by the speed of light
- COMPARE TO PHOTONS
- Photon energy = ℏω
- Photon momentum = ℏω/c = ℏk
- In vector terms photon momentum = ℏ**k**
- Total energy in the beam = N ℏω

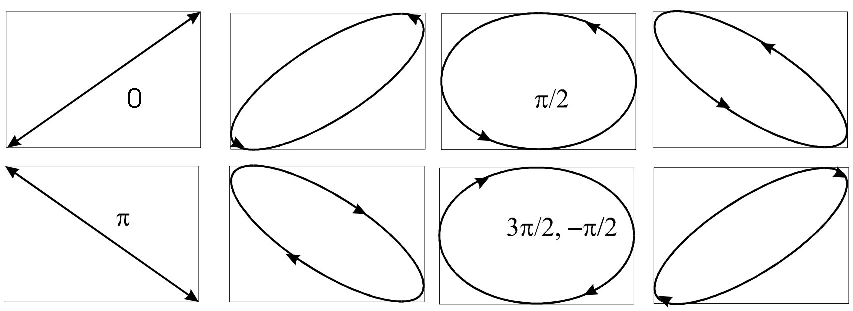
Polarisation

- If light is a vector wave with an electric field and a magnetic field orthogonal to one another
- Then we can resolve any single wave propagating along the z axis into two parts with electric vectors along the x and y axes respectively - E_x and E_y
- BUT it is possible that these are not in phase -there is a specific phase difference between the two waves
- Restrictions...
- Not just any pair of fields will do
- MUST have same frequency (exactly) and a defined phase relationship between then
- OBSERVABLE features can have many frequencies BUT must have a defined phase relationship which applies for all frequencies
- OBSERVABLE is formed of many sub-parts each of which is of a single frequency and defined phase relationship

Some Common Polarisations

- 0, mπ - total electric vector is a straight line at an angle given by $\tan^{-1}(E_y/E_x)$ - linearly polarised
- $(2m+1)\pi/2$ and $E_y = E_x$ - circularly polarised
- Anything else - elliptically polarised light
- All described by the locus of the vector
- $E_x \cos\omega t\ \mathbf{x} + E_y \cos(\omega t+\delta)\ \mathbf{y}$
- Or in complex notation
  $E_x \exp i\omega t\ \mathbf{x} + E_y \exp i(\omega t+\delta)\ \mathbf{y} \rightarrow E_x\ \mathbf{x} + E_y \exp i\delta\ \mathbf{y}$

Some Common Polarisations

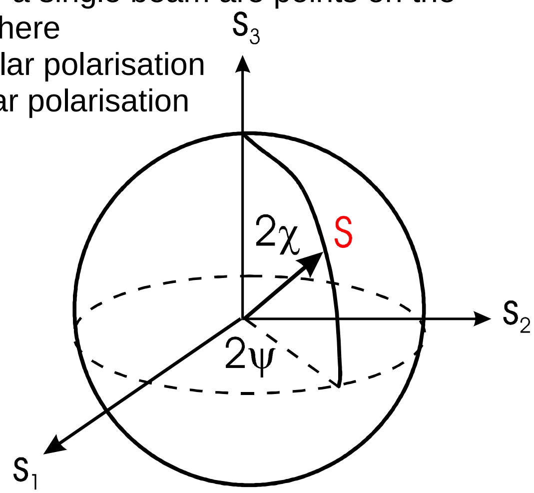


But You Can Only Measure Energy

- OK so we must characterise things in terms of energy quantities
- $s_o = E_{ox}^2 + E_{oy}^2$ $= s_0$ = Total energy
- $s_1 = E_{ox}^2 - E_{oy}^2$ $= s_0 \sin2\chi\cos2\psi$ = Difference in energy
- $s_2 = 2 E_{ox} E_{oy} \cos(-\delta)$ $= s_0 \sin2\chi\sin2\psi$
- $s_3 = 2 E_{ox} E_{oy} \sin(-\delta)$ $= s_0 \cos2\chi$
- Stokes Parameters - after Lord Stokes of course!

Poincaré Sphere

- $s_0^2 = s_1^2 + s_2^2 + s_3^2$
- The angles 2χ and 2ψ form the latitude and longitude on the sphere
- All polarisations of a single beam are points on the surfaces of the sphere
- Eg poles are circular polarisation
- Eg equator is linear polarisation



The Real World

- See a sum of lots of little "wavelets" (except lasers!)
- Stokes parameters are energy-like quantities and can be added up for the sum of a set of "wavelets"
- $s_0 = \langle E_{ox}^2 \rangle + \langle E_{oy}^2 \rangle \quad = s_0$
- $s_1 = \langle E_{ox}^2 \rangle - \langle E_{oy}^2 \rangle \quad = s_0 \langle \sin 2\chi \cos 2\psi \rangle$
- $s_2 = 2 \langle E_{ox} E_{oy} \cos(-\delta) \rangle \quad = s_0 \langle \sin 2\chi \sin 2\psi \rangle$
- $s_3 = 2 \langle E_{ox} E_{oy} \sin(-\delta) \rangle \quad = s_0 \langle \cos 2\chi \rangle$

Natural Light

- "Natural" light does not follow the restrictions on frequency and phase as previous discussion
- "Natural" light considered as random superposition of "wavelets" of random phasing
- $s_0^2 \geq s_1^2 + s_2^2 + s_3^2$
- For light from a thermal source
- $s_0 <> 0, \; s_1 = s_2 = s_3 = 0$
- For white light from a linear polariser
- $s_0 = \pm s_1 <> 0, \; s_2 = s_3 = 0$

Radiometric Units

- Defined by history - not by logic!!
- Energy is the fundamental unit (J)
- Power is energy per unit time (W)
- Power per unit solid angle
  - "Pointance" - W sr^{-1} - intensity
- Power per unit area
  - "areance" - W m^{-2} - ??????
- Power per unit area per unit solid angle
  - "sterance" - W m^{-2} sr^{-1} - radiance

Depends Upon Your Point of View

- For sources interested in...
  - Total power (W)
  - Power into a solid angle (Wsr^{-1})
- For receivers interested in....
  - Total power per unit area (Wm^{-2})
  - Power per unit area per unit solid angle ($Wm^{-2}sr^{-1}$)