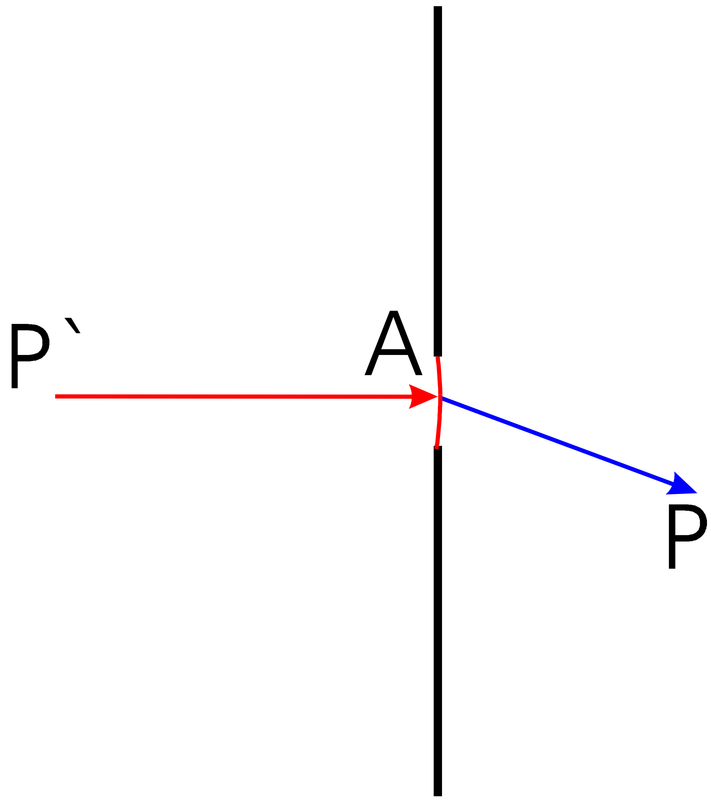
Lecture 18

Fresnel Diffraction

It's Still Too Complicated!!

- Scalar wave equation applied to a screen with a hole in it

$$U_P = \frac{-ikU_0}{4\pi}\int_A \frac{e^{ik(r+r^/)}}{rr^/}\left(\cos(n,r) - \cos(n,r^/)\right)\, dS$$



- This is the Fresnel-Kirchoff integral formula
- Shows a phase factor of -i for the diffracted wave
- Integration of secondary radiators across the aperture with an obliquity factor

The Huygen’s-Kirchoff Formula

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r + r')}}{rr'} \left(\cos(n,r) - \cos(n,r') \right) dS$$

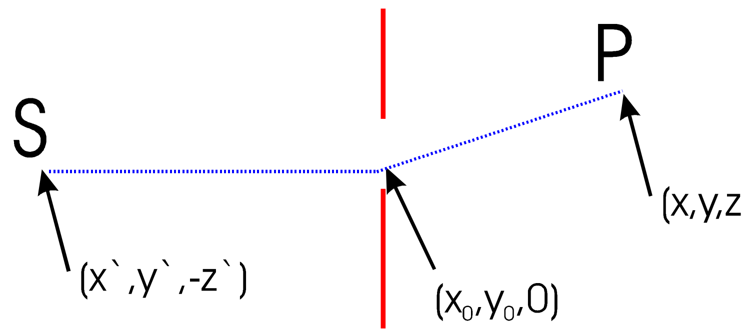
- Has three essential terms
  - exp(ikr‘)/r‘ - the amplitude of the wave at the aperture (point source is assumed here - you may have to integrate for an extended source)
  - exp(ikr)/r - a spherical wave from each point on the aperture (Huygen’s secondary radiators)
  - cos(n,r) - cos(n,r`) - the obliquity factor - often denoted as Q(r,r`) or Q(P,P`)

The Huygen's-Kirchoff Formula

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r + r^{/})}}{rr^{/}} \left(\cos(n,r) - \cos(n,r^{/}) \right) dS$$

- For many cases kr, kr‘ are very large
  - Often possible to approximate r, r‘ by simple formulae
  - BUT the variation in the exponent is much more sensitive than the variation in the denominator
- Also in many cases all the angles are small
  - If all the angles are small then cos(n,r) - cos(n,r‘) $\approx$ 2

The Huygen's-Kirchoff Formula



- Any one of the diagonal distances is given by Pythagoras

$$r = \sqrt{z^2 + (x - x_0)^2 + (y - y_0)^2}$$

$$= z\left(1 + \frac{(x - x_0)^2}{z^2} + \frac{(y - y_0)^2}{z^2}\right)^{1/2}$$

$$\simeq z + \frac{(x - x_0)^2}{2z} + \frac{(y - y_0)^2}{2z}$$

- Last one only true for "small" $x-x_0$, $y-y_0$

The Fresnel Formula

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r + r^/)}}{rr^/} \left(\cos(n,r) - \cos(n,r^/) \right) dS$$

$$= \frac{-ikU_0}{2\pi zz^/} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} \left[(x_0 - x_m)^2 + (y_0 - y_m)^2 \right] \right) dS$$

- where PS is the source-observation point distance
- $x_m \equiv (zx' + z'x)/(z+z')$ $x_m \rightarrow x$ as $z' \rightarrow \infty$
- $y_m \equiv (zy' + z'y)/(z+z')$ $y_m \rightarrow y$ as $z' \rightarrow \infty$
- $z_a \equiv (zz')/(z+z')$ $z_a \rightarrow z$ as $z' \rightarrow \infty$

The Fresnel Formula

$$U_P = \frac{-ikU_0}{2\pi z z'} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} \left[(x_0 - x_m)^2 + (y_0 - y_m)^2 \right] \right) dS$$

- let z' → ∞ and use a point on the axis x,y = 0

$$U_P = B \int_A \exp\left(\frac{ik}{2z} \left[x_0^2 + y_0^2 \right] \right) dS$$

$$= B \int_{x_1}^{x_2} \exp\left(\frac{ikx_0^2}{2z} \right) dx_0 \int_{y_1}^{y_2} \exp\left(\frac{iky_0^2}{2z} \right) dy_0$$

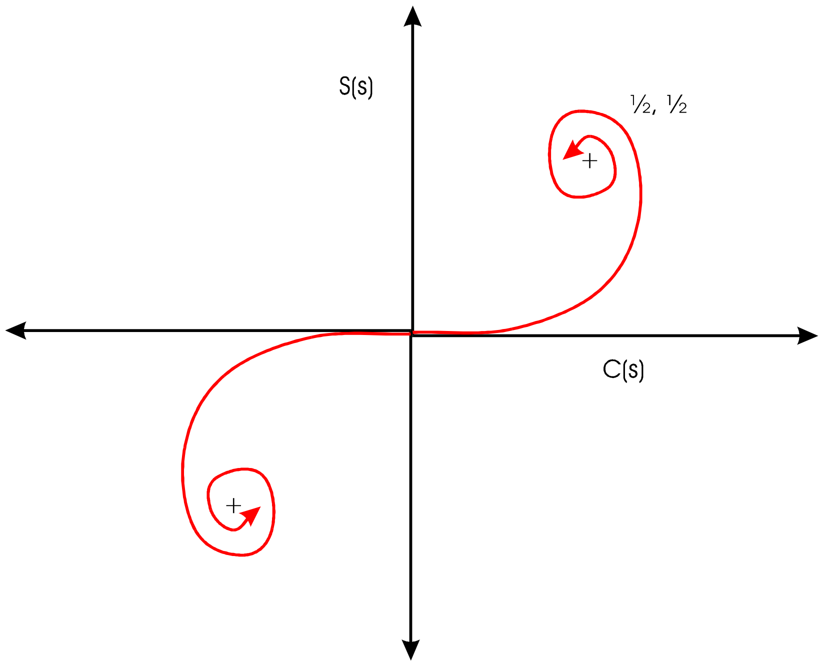
$$= B \int_{x_1}^{x_2} \exp\left(\frac{i\pi u^2}{2} \right) dx_0 \int_{y_1}^{y_2} \exp\left(\frac{i\pi v^2}{2} \right) dy_0$$

- where $u^2 = kx_0^2/(\pi z)$, $v^2 = ky_0^2/(\pi z)$

The Fresnel Integral

$$\int_{s_1}^{s_2} \exp(i\pi w^2/2)\,dw = \int_{s_1}^{s_2} \cos(\pi w^2/2)\,dw + i\int_{s_1}^{s_2} \sin(\pi w^2/2)\,dw$$
$$= C(s) + iS(s)$$

- Cornu spiral
- As s →± ∞, C(s),S(s)→ ± ½
- Value of integral can be evaluated numerically

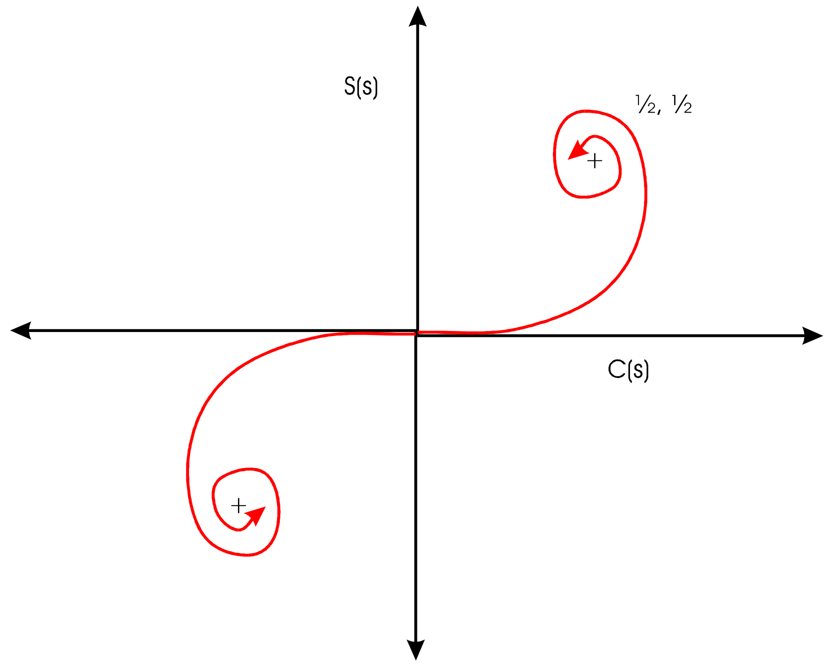


The Fresnel Integral

$$U_P = B \int_{x_1}^{x_2} \exp\left(\frac{i\pi u^2}{2}\right) dx \int_{y_1}^{y_2} \exp\left(\frac{i\pi v^2}{2}\right) dy$$

$$= \frac{U_{P_0}}{(1+i)^2} \left[C(s) + iS(s)\right]_{x_1}^{x_2} \left[C(s) + iS(s)\right]_{y_1}^{y_2}$$

- If limits are infinity the signal must be U_{P0} [normalization]
- Now take in infinitely long slit
  - eliminates y_0 terms
- Take a single edge at $x_0 = x$

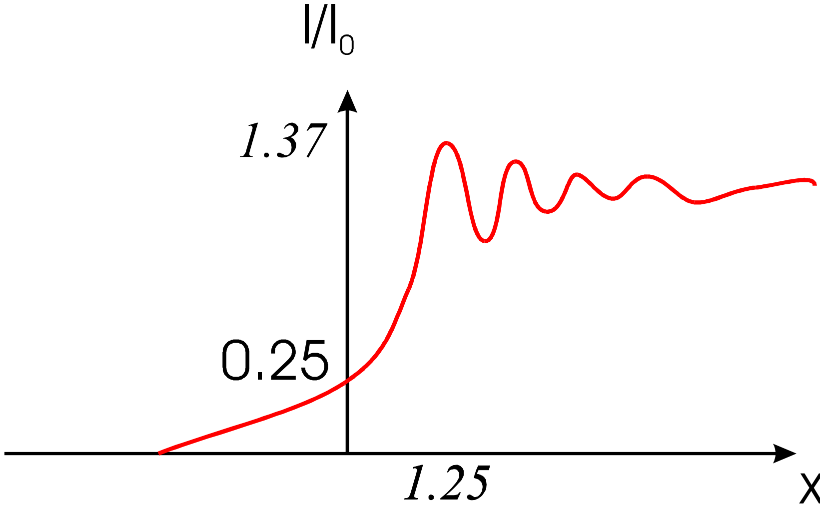


The Fresnel Integral

$$U_P = \frac{U_{P_0}}{(1+i)^2}\left[C(s)+iS(s)\right]_{x_1}^{x_2}\left[C(s)+iS(s)\right]_{y_1}^{y_2}$$

$$= \frac{U_{P_0}}{(1+i)}\left(C(x)+iS(x)+\frac{1}{2}+\frac{1}{2}i\right)$$

- At x = 0, $U_P = U_{P0}/2$ which implies 0.25 Intensity
- Moving x is equivalent to moving observation point since everything else is (semi-)infinite



The Circular Aperture

$$U_P = \frac{-ikU_0}{4\pi} \int_A \frac{e^{ik(r+r')}}{rr'} \left(\cos(n,r) - \cos(n,r') \right) dS$$

$$= \frac{-ikU_0}{2\pi zz'} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} (r_0 - r_m)^2 \right) Q \; dS$$

- Need Obliquity Factor to save us from a fate worse than death (an oscillating integral!!)

The Circular Aperture

- If $r_m = 0$, then we can recast this problem in circular symmetry in terms of $\psi = kr_0^2/(2z_a)$ and U_{P0} the undisturbed wave

$$U_P = -iU_{P_0} \int_0^{\psi_0} \exp(i\psi)\ Q(\psi)\ d\psi$$

- $\psi_0 = kR^2/2z$
- This integral oscillates unless $Q(\psi) \rightarrow 0$ as $\psi \rightarrow \infty$, then it collapses to a value of i