Lecture 20

Fraunhofer Diffraction - Transforms

Fraunhofer Diffraction

$$U_P = \frac{-ikU_0}{2\pi zz'} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} \left[(x_0 - x_m)^2 + (y_0 - y_m)^2\right]\right) dS$$

$$= \frac{-ikU_0}{2\pi zz'} e^{ik|PS|} \exp\left(\frac{ik}{2z_a} [x_m^2 + y_m^2] \right)$$

$$. \int_A \exp\left(\frac{ik}{z_a} \left[(x_0 x_m + y_0 y_m\right]\right) dx_0 dy_0$$

- This is effectively saying that BOTH the source and the observation point are “far” from the aperture

Fraunhofer Diffraction

- If we now generalise this by replacing the plane wave from the source (U_0/z') by a general distribution across the aperture $U(x_0,y_0)$
- Recognise that in a paraxial approximation a long way from the aperture it is the ANGLES that matter, not the positions
  - $u = kx_m/z_a$, $v = ky_m/z_a$

$$U_P = \frac{-ikU_0}{2\pi z z'} e^{ik|PS|} \exp\left(\frac{ik}{2z_a} [x_m^2 + y_m^2] \right)$$

$$. \quad \int_A \exp\left(\frac{ik}{z_a} \left[(x_0 x_m + y_0 y_m \right] \right) dx_0 dy_0$$

$$= \text{MOC\&FF} \int_A U_0(x_0,y_0) \exp(-i\,[ux_0 + vy_0]\,)\, dx_0 dy_0$$

- In case you don't recognise it - the final expression is a 2-

D Fourier Transform relation!

Fraunhofer Diffraction

$$U_P = \frac{-ik}{2\pi z} e^{ik|PS|} \exp\left(\frac{iz_a}{2k} [u^2 + v^2] \right)$$

$$\cdot \int_A U_0(x_0, y_0) \exp(-i [ux_0 + vy_0]) \, dx_0 dy_0$$

- relates x_0, y_0 space to u, v space
- e.g. for a square aperture and a plane wave -

$$U_P = \frac{-ik}{2\pi z} \exp\left(\frac{iz}{2k} [u^2 + v^2] \right) U_a \ell_x \ell_y sinc\left(\frac{\ell_x u}{2\pi} \right) sinc\left(\frac{\ell_y v}{2\pi} \right)$$

- the intensity is the square of this function
- For a circular function it is almost the same except that it's a Bessel function, not a sinc function

Fraunhofer Diffraction

$$U_P = \frac{-ik}{2\pi z} e^{ik|PS|} \exp\left(\frac{iz_a}{2k} [u^2 + v^2] \right)$$

$$\cdot \int_A U_0(x_0, y_0) \exp(-i [ux_0 + vy_0]) \, dx_0 dy_0$$

- If this is so then we have a powerful technique
  - Applies for large distances
  - Applies for paraxial approximation
  - Maybe could relieve those with thought
  - BUT any $U_0(x_0,y_0)$ can be used
  - We have computers - Have FFT - will compute!

Slits and Gratings

- Consider a slit of dimension a,b at origin in x_0, y_0 plane with light incident at angle θ
  - Wavefront is $\exp(ikx_0\sin\theta)\mathrm{rect}(x_0/a)\mathrm{rect}(y_0/b)$
  - Also called the "aperture function"
- Now consider a whole set of slits (a grating)
  - use a convolution
  - Wavefront is
    $\exp(ikx_0\sin\theta)\mathrm{rect}(x_0/a)\mathrm{rect}(y_0/b)*\mathrm{comb}(x_0/\Lambda)$
- But that's infinite - need it finite
  - $\exp(ikx_0\sin\theta)\mathrm{rect}(x_0/a)\mathrm{rect}(y_0/b)*$
    $[\mathrm{comb}(x_0/\Lambda)\mathrm{rect}(x_0/(n\Lambda))]$
- For the angular distribution - just need the Fourier Transform
- Assume b infinite

Slits and Gratings

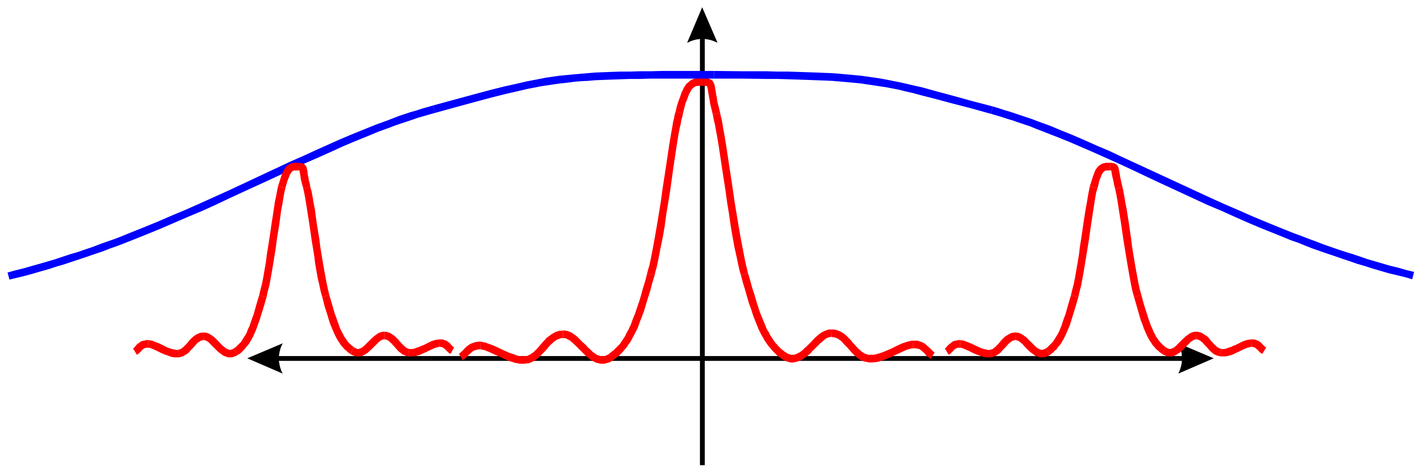
- $\mathscr{F}\{\exp(ikx_0\sin\theta)\mathrm{rect}(x_0/a) * [\mathrm{comb}(x_0/\Lambda)\mathrm{rect}(x_0/(n\Lambda))]\}$
- $\mathscr{F}\{\exp(ikx_0\sin\theta)\mathrm{rect}(x_0/a)\ \mathscr{F}\{[\mathrm{comb}(x_0/\Lambda)\mathrm{rect}(x_0/(n\Lambda))]\}$
- Use shift property + “six transforms” on first term
- $|a|\ \mathrm{sinc}(\ a[u + k\sin\theta]/2\ )\ \mathscr{F}\{[\mathrm{comb}(x_0/\Lambda)\mathrm{rect}(x_0/(n\Lambda))]\}$
- Second term is a finite sum of δ function

$$\mathscr{F}\left\{ \sum_{n=-(N-1)/2}^{(N-1)/2} \delta(x_0 - n\Lambda) \right\} = \sum_{n=-(N-1)/2}^{(N-1)/2} \exp(-inu\Lambda)$$

- and the total result becomes

$$|a|\ \mathrm{sinc}\left(\pi[\sin\theta + \sin\theta_i]\ a/\lambda\right) \frac{\sin\left(\dfrac{\pi N\Lambda}{\lambda}[\sin\theta + \sin\theta_i]\right)}{\sin\left(\dfrac{\pi\Lambda}{\lambda}[\sin\theta + \sin\theta_i]\right)}$$

Grating Resolving Power



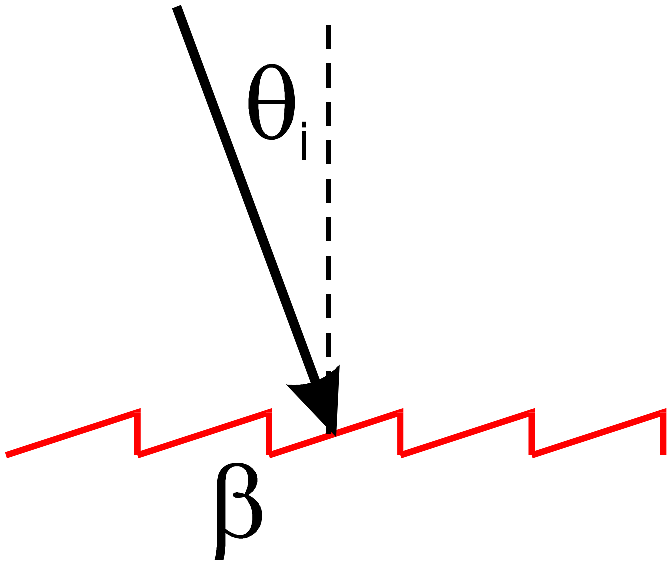
- Peaks are from sin ratio (squared for intensity)
- Envelope is from the sinc function
- For normal incidence close to axis θ_i =0 then width of peak is of order $(\pi N\Lambda/\lambda)\cos\theta\delta\theta = \pi$
- Maxima occur at $(\pi\Lambda/\lambda)\sin\theta = m\pi$
  - $\Lambda\cos\theta\delta\theta = m\delta\lambda$
- combining these $\lambda/\delta\lambda = mN$
  - at 500 slits/mm and 50mm grating
  - resolving power is 2.5×10^4 if m = 1

Grating Energy Split

$$|a| \ \mathrm{sinc}\left(\pi\,[\sin\theta + \sin\theta_i]\,a/\lambda\right) \frac{\sin\left(\dfrac{\pi N\Lambda}{\lambda}[\sin\theta + \sin\theta_i]\right)}{\sin\left(\dfrac{\pi\Lambda}{\lambda}[\sin\theta + \sin\theta_i]\right)}$$

- This is the product of the FT of
  - the slit (broad)
  - the repetitive structure (narrow)
- The slit peaks the energy in the forward direction m=0 where there is no dispersion (resolving power)!!! - bad idea!!
- Playing with the slit might permit us to move the peak energy from straight on (or straight back for reflection) to another order

Grating Energy Split



- Suppose we use a (reflection) grating for which the slit "height" is $h = h_0 x_0 \tan\beta$ - ie a sawtooth
- Phase delay is composed of two parts
  - phase delay of "tilted" plane wave - $kx_0 \sin\theta_i$
  - phase delay due to sawtooth - $h\cos\theta_i$
- Total phase delay is the sum of these and therefore result is

Grating Energy Split

$$\mathrm{sinc}\left(\pi\,[\sin\theta + \sin\theta_i + \tan\beta\cos\theta_i]\,\Lambda/\lambda\right)\;\frac{\sin\left(\dfrac{\pi N\Lambda}{\lambda}[\sin\theta + \sin\theta_i]\right)}{\sin\left(\dfrac{\pi\Lambda}{\lambda}[\sin\theta + \sin\theta_i]\right)}$$

- Notice that if $\beta = 0$ - grating disappears!!
- if $m\lambda/\Lambda + \sin\theta_i + \cos\theta_i \tan\beta = 0$ the power peaks in order m

A Lens as an Aperture

- If we compute the phase delay across a lens we can make it an aperture function!
- Take a plane before the lens and a plane after
- Can be shown that in the paraxial approximation the thickness of the material is...

$$\Delta(x_0,y_0) = \Delta_0 - \frac{x_0^2 + y_0^2}{2}\left[1/R_1 - 1/R_2\right]$$

- and the phase delay is given by

$$\varphi = kn\Delta_0 - \frac{k}{2f}\left[x_0^2 + y_0^2\right]$$

A Lens as an Aperture

- Take the "pupil function" (ie physical extent of the lens) as $P(x_0,y_0)$ - almost always a cylinder function
- Take the incoming wavefront as $t(x_0,y_0)$ - e.g. plane wave
- Take the phase manipulation of the lens
- Take the z dimension to be the focal length - f
- Ignore phase factors and quadratic terms

$$\int\int_{-\infty}^{+\infty} P(x_0,y_0)\ t(x_0,y_0)\ \exp\left(-\frac{2\pi i}{f\lambda}\left[x_0x+y_0y\right]\right)\ dx_0dy_0$$

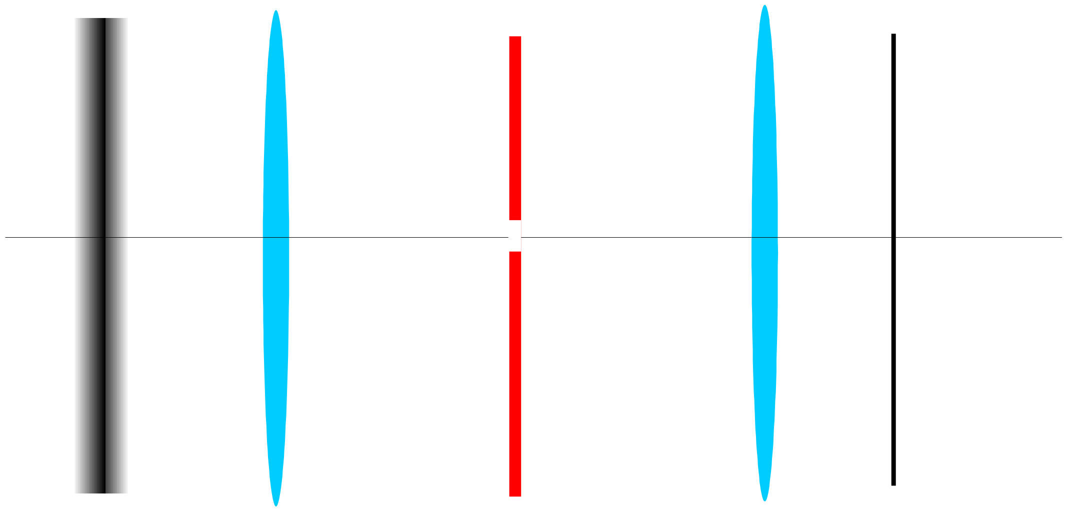
- Its a Fourier transform again!!
- Focal plane image is FT of aperture function in front of lens

A Lens as an Aperture

$$\iint_{-\infty}^{+\infty} P(x_0,y_0)\ \ t(x_0,y_0)\ \ \exp\left(-\frac{2\pi i}{f\lambda}\left[x_0 x + y_0 y\right]\right)\ \ dx_0 dy_0$$

- if input wave is plane and P is a cylinder function cyl(r/R)
- Image plane is an airy function of size 2 (1.22λf/(2R))
- Or using f/(2R) as the *f-number* - 2.44λf^*
- So if λ = 500nm and f^* of order 1 image size is of order 1μm
- Generally f^* > 1 so spot size is > 1μm

A Lens as an Aperture



- In the formula above
  - low spatial frequencies in incoming wave turn up near centre
  - high spatial frequencies turn up away from the centre
- By putting an aperture at the focal point we can “clean up” a mucky plane wave - only allow low frequency components through