Lecture 22

Gaussian Beams
Paraxial or Perturbation Analysis

Perturbation Analysis

- Many of the examples we have treated have phase which
  - varies fast in the DOP (z)
  - varies slowly as a function of the other co-ordinates (x,y)
- Fresnel diffraction varies quadratically in the x,y direction
  - (1/r)exp(ikr) -> (1/z)exp(ikz) exp(ik(x^2+y^2)/(2z))
  - separates the fast-changing z-direction from the slow changing x,y direction
- Can we generalise? (Yes, or I wouldn't be doing this...)
- write the wave as u(x,y,z)exp(ikz) - subst in wave equation

$$\nabla_T^2 u + \frac{\delta^2 u}{\delta z^2} + 2ik\frac{\delta u}{\delta z} = 0$$

- ∇_T is the 2-D grad function

Paraxial Wave Equation

- If we now assume that the function
  - varies slowly in z on the scale of a wavelength $|\partial u/\partial z| \ll k|u|$
  - that it is smooth - higher order differentials can be ignored

$$\nabla_T^2 u + \frac{\delta^2 u}{\delta z^2} + 2ik\frac{\delta u}{\delta z} \rightarrow \nabla_T^2 u + 2ik\frac{\delta u}{\delta z} = 0$$

- The *slowly varying envelope approximation (SVEA)* leads to the *paraxial wave equation*
- Note that $(1/z)\exp(ik(x^2+y^2)/(2z))$ is a solution of the above equation
- Trouble with that solution is that it has infinite extent - is there a similar solution which has finite extent - ie looks something like a pencil beam?

A Gaussian Beam

- Yes, there is....
  - $(1/z)\exp(ik(x^2+y^2)/(2z))$ ->
  - $u_{00}'(x,y,z) = (1/(z-iz_0))\exp(ik(x^2+y^2)/(2(z-iz_0))$
  - $z \to z - iz_0$
- Normalise over x, y

$$u_{00}(x,y,z) = \sqrt{\frac{kz_0}{\pi}}\,\frac{1}{z-iz_0}\,\exp\left(\frac{ik[x^2+y^2]}{2(z-iz_0)}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\,\exp\left(-\frac{[x^2+y^2]}{w^2}\right)\exp\left(\frac{ik[x^2+y^2]}{2R}\right)$$

- $w^2(z) = w_0^2(1 + z^2/z_0^2)$, $w_0^2 = 2z_0/k$
- $R = (z^2+z_0^2)/z$, $\tan\varphi = z/z_0$

Properties of the Gaussian Beam

$$u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\,\exp\left(-\frac{r^2}{w^2}\right)\,\exp\left(\frac{ikr^2}{2R}\right)$$

- *Fundamental Gaussian Beam Solution*
- Function of one parameter - z_0
- Circularly symmetric - function of r
- Gaussian extent transversely - w is the e^{-1} point of amplitude
- at z = 0, w = w_0 and is the minimum extent of beam
- at z = z_0, w = $\sqrt{2}\,w_0$ - z_0 is called the *confocal parameter*

Properties of the Gaussian Beam

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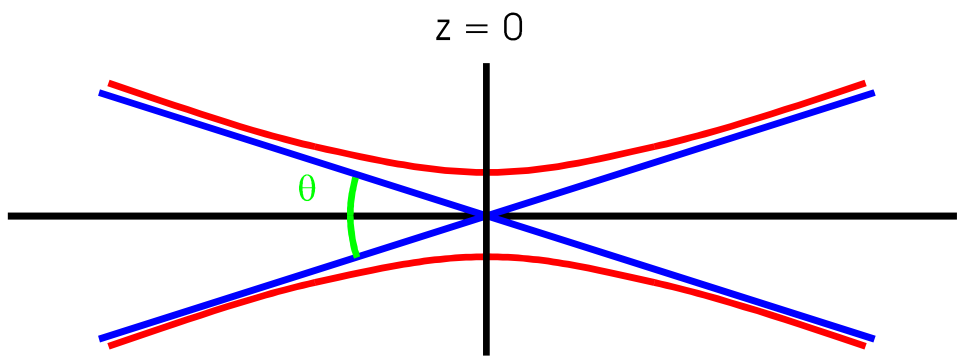
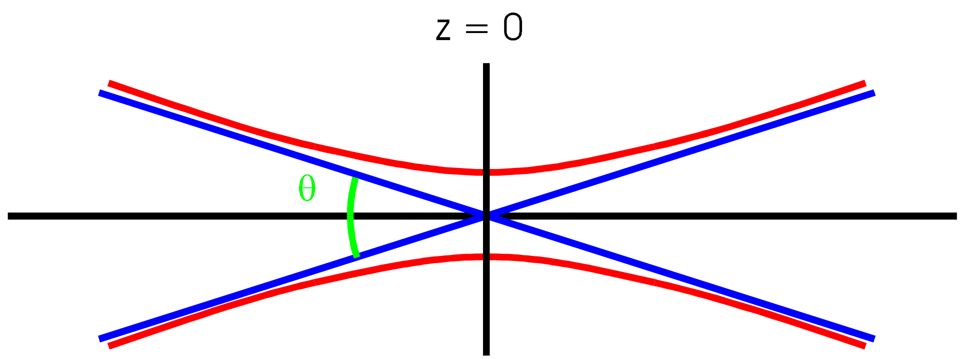


- What is the shape of the red line of constant amplitude?
- $r^2/w^2 = C$
- $r^2 - (Cw_0^2/z_0^2)z^2 = Cw_0^2$
- Hyberolas
- Note that the confocal parameter z_0 gives the distance over which the beam is "sort of" collimated and w_0 gives the minimum beam size
- Since $z_0 = kw_0^2/2$ you can't have a small beam collimated for a long distance. - small waist, large divergence and vv

Properties of the Gaussian Beam

$$u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\exp\left(-\frac{r^2}{w^2}\right)\exp\left(\frac{ikr^2}{2R}\right)$$

- What is the asymptotic angle θ?
- for large r,z,
  $\tan(\theta/2) = w/z$
  $= w_0(z/z_0)/z$
  $= w_0/z_0 = 2/(kw_0)$
- Putting in numbers at 500nm and a 1mm spot θ = 0.32mr
- ie spot size increases by 1mm/3m - laser-beam like



Properties of the Gaussian Beam

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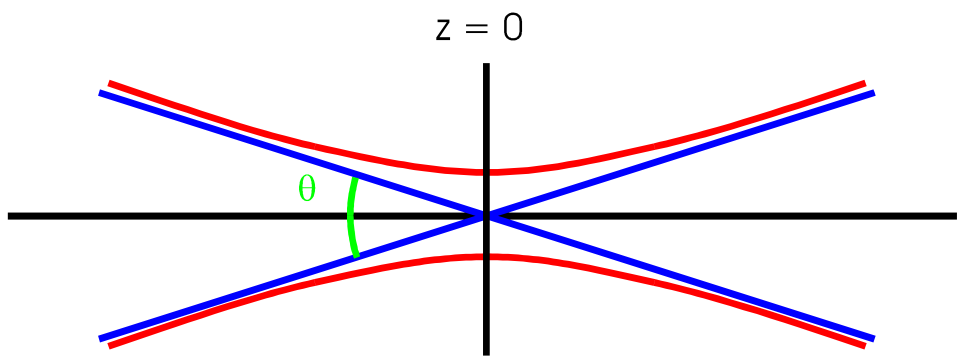
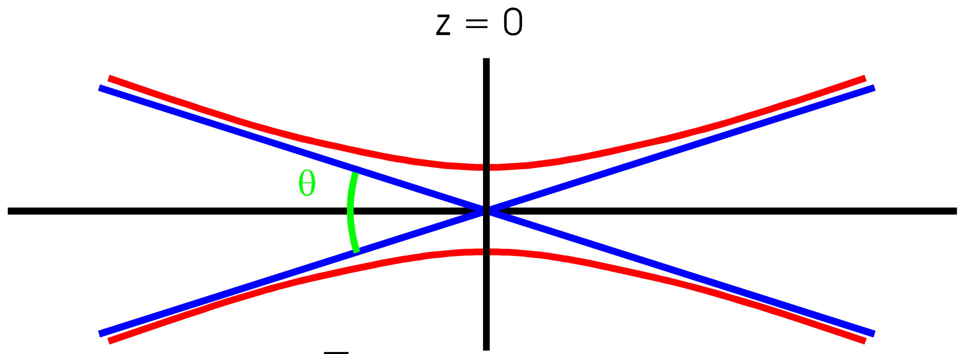


- What is the significance of R?
- R is the radius of curvature of the surfaces of constant phase
- as z → 0, R → ∞
- as z → ∞, R → z
- What is the significance of φ?
  - φ is related to the velocity of surfaces of constant phase
  - Remember this isn't a plane wave and doesn't even satisfy the full wave equation

Properties of the Gaussian Beam

$$u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\,\exp\left(-\frac{r^2}{w^2}\right)\exp\left(\frac{ikr^2}{2R}\right)$$

- Beam propagates as exp(ikz-iφ)
- If we take k_{eff} as the average over 0 → z



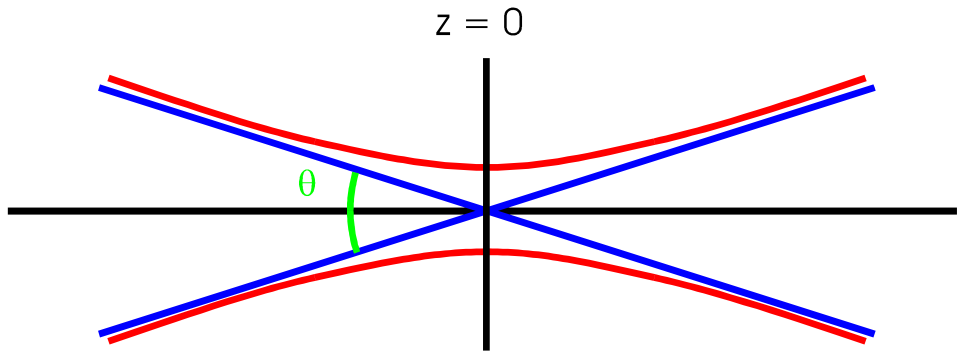
$$k_{eff} = k - \frac{d\varphi}{dz} = \frac{\omega}{c} - \frac{z_0}{z^2 + z_0^2} < \frac{\omega}{c}$$

- which implies that the phase velocity is > c which isn't a problem because this isn't a solution of the full wave equation

Properties of the Gaussian Beam

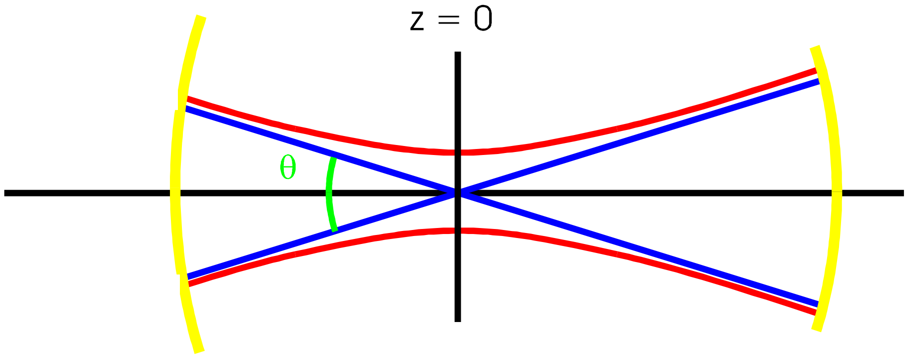
$$u_{00}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{w}\,e^{-i\varphi}\,\exp\left(-\frac{r^2}{w^2}\right)\,\exp\left(\frac{ikr^2}{2R}\right)$$

- When is all this valid?
- When $|\partial u/\partial z| \ll k|u|$
- which turns out to be when $1/z_0 \ll k$ or $\lambda \ll 1.4\pi w_0$



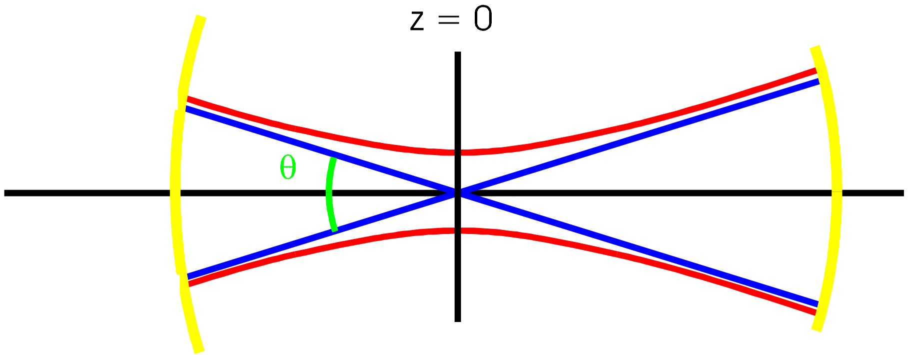
- Beam waist much larger than wavelength

Standing Waves In a 1.5-D cavity



- A cavity is formed by two mirrors assumed for simplicity to be 100% reflecting
- A mode (or standing wave) happens if...
  - Amplitude of field at any point is stationary
  - Phase is also stationary
- Neglect refraction losses
- Can set up a mode if beam is Gaussian and mirrors match R_1,R_2 the radii of the wavefronts at each end

Standing Waves In a 1.5-D cavity

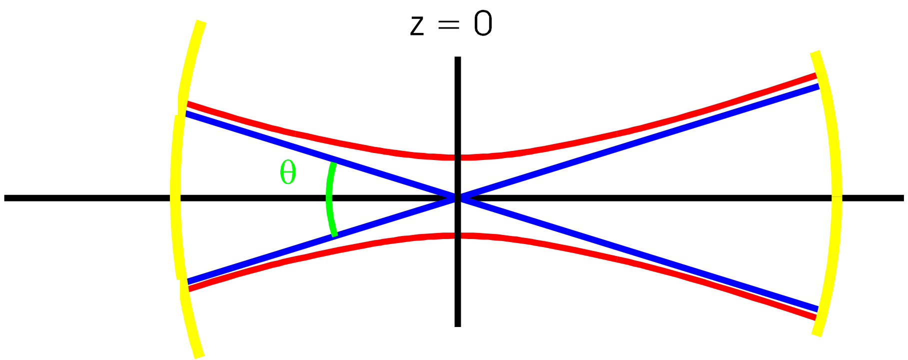


- $-R_1 = (z_1^{\,2}+z_0^{\,2})/z$
- $R_2 = (z_2^{\,2}+z_0^{\,2})/z$
- Remember one z has to be -ve
- Let $z_2 - z_1 = L$
- Let $g_{1,2} = (1 - L/R_{1,2})$
- Solve for z_0

$$z_0^2 = \frac{L^2 g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2g_1 g_2)^2}, \qquad w_0^2 = \frac{2}{k}\,\frac{L\sqrt{g_1 g_2 (1 - g_1 g_2)}}{(g_1 + g_2 - 2g_1 g_2)}$$

Standing Waves In a 1.5-D cavity

- and for the spot sizes on the mirror $w_{1,2}$

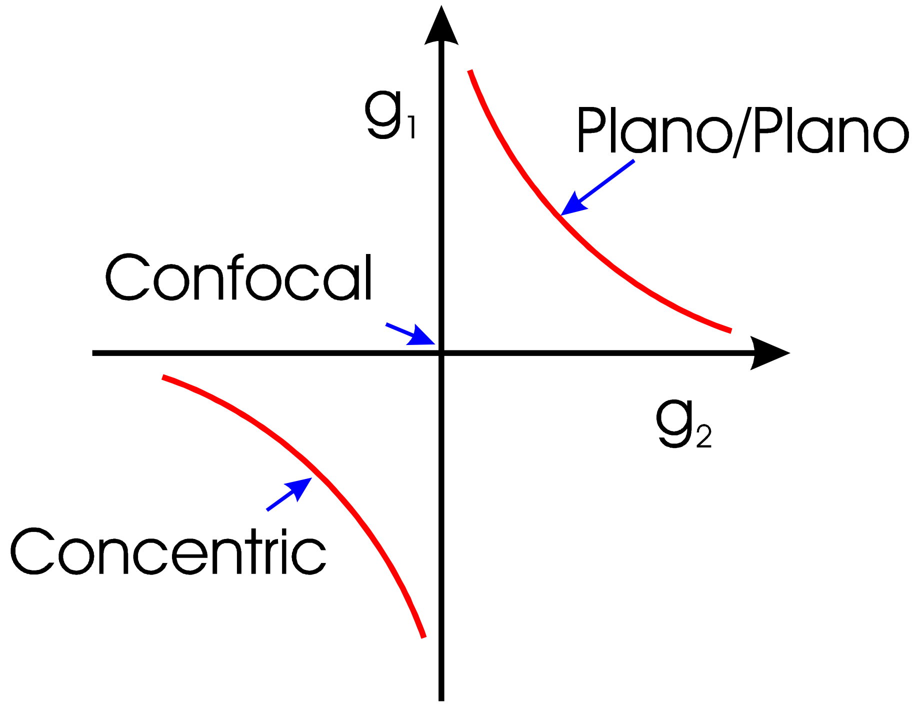


$$w_{1,2}^2 = \frac{L\lambda}{\pi}\sqrt{\frac{g_{2,1}}{g_{1,2}(1-g_1g_2)}}$$

- These solutions exist if $0 < g_1g_2 < 1$
- Stable solutions require all of w_1, w_2 and w_0 to be finite and larger than the wavelength λ

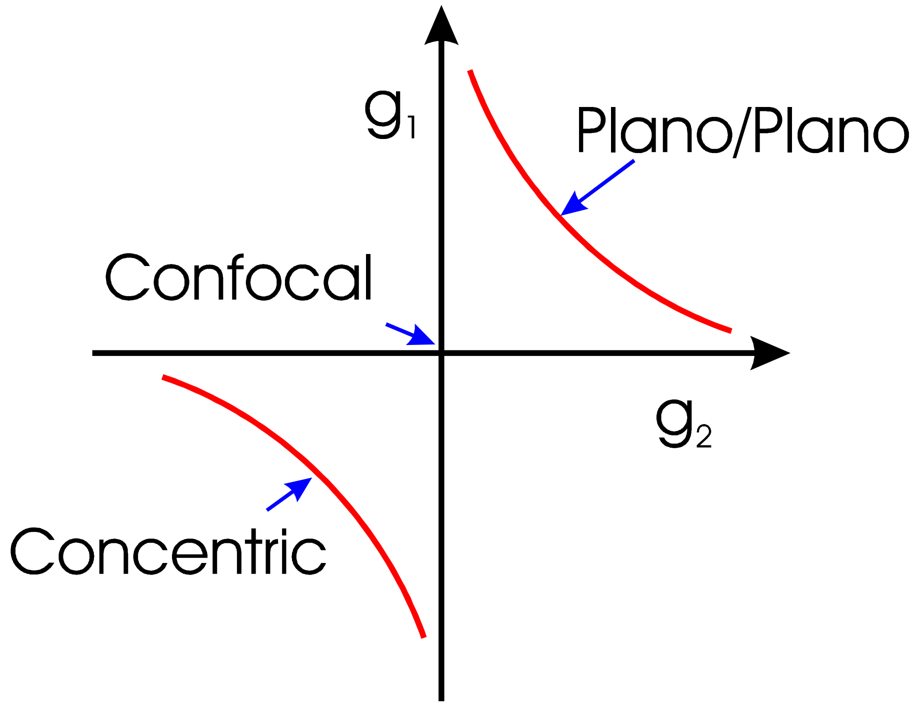
Standing Waves In a 1.5-D cavity

- If $g_1 = g_2 = 1$ both mirrors are plane - FP
  - all w are infinite!
- If $g_1 = g_2 = 0$, R = L and the cavity is confocal
  - OK but $w_0 = 0$
- If $g_1 = g_2 = -1$, R = L/2 the cavity is concentric
  - $w_{1,2}$ are infinite!



Standing Waves In a 1.5-D cavity

- if $g_1 = 0$, $g_2 < 1$ we have one flat mirror at the beam waist and one curved mirror



$$w_{0,1}^2 = \frac{2L}{k}\sqrt{\frac{g_2}{1-g_2}}, \quad w_2^2 = \frac{2L}{k}\sqrt{\frac{1}{g_2(1-g_2)}}$$

- This is OK for a range of g_2
- For L = 1m, λ = 1μm for R_2 = 5m-20m $w_{0,1,2}$ of order 1mm