PREDICTING THE FUTURE USING PHYSICS

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I. Introduction

These are supplementary notes for a brief discussion in PHY131, a 1st-year physics course primarily for students in the life sciences.

We begin by discussing more fully something that has been implied throughout the term: that the classical physics description of the universe allows us to predict the future. The topic of *chaos* will be central to the discussion, and we will extend our understanding to more general features of chaotic systems. We will then discuss cases where classical physics is largely incapable of such predictions.

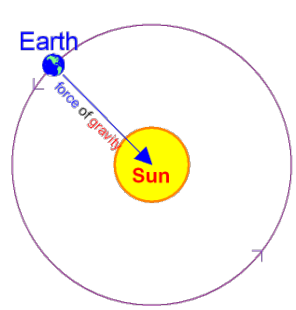
Below, our explorations will involve a number of animations, some using Flash. Thus you will want to have access to a computer with the Flash player and that is connected to the internet while reading these notes. Sadly, because of a bad case of "not invented here" syndrome at Apple, the Flash player is not available for Apple iPads and iPhones.

II. Predicting the Future

One of Newton's many achievements was his theory of gravitation. If 2 masses, *m* and *M*, are separated by a distance *r*, then the attractive gravitational force each exerts on the other is given by:

$$F = G\frac{mM}{r^2} \quad (1)$$

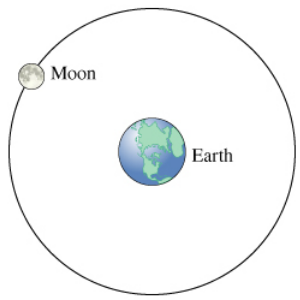
where *G* is called the *universal gravitational constant*.



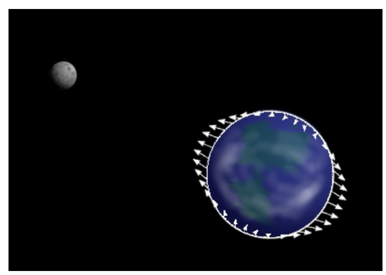
If we imagine a "solar system" that contains just a Sun fixed in space and an Earth free to move, as in the figure to the right, then Newton's Law of Gravitation, Eqn. 1, can be coupled with his 2nd Law of Motion, $\vec{a} = \vec{F}/m$ or $\vec{F} = m\vec{a}$, to show that the motion of the Earth around the Sun is an ellipse with the Sun at one focus of the ellipse. It turns out the orbit of the real Earth is almost a perfect circle, which is just another type of ellipse. In this course, it would not have taken much time to show in detail how to find this solution, although we have not done so.

Soon it will be important to note that assuming that the "solar system" has only one planet, the Earth, means we are ignoring any gravitational forces on Earth due to other objects. This is, of course, not the real situation, since the other planets and the Moon exert comparatively small but non-zero gravitational forces on the Earth. In the 19th century physicists learned how to deal with these extra forces as small perturbations of the elliptical orbit.

If we think about the Moon in orbit around the Earth, as shown to the right, then if we approximate that the Earth is fixed in space, then the Moon's orbit will similarly be an ellipse with the Earth at one focus of the ellipse. Just as for the Earth-Sun system, we are assuming that any gravitational forces on the Moon due to the Sun and other planets are negligible.



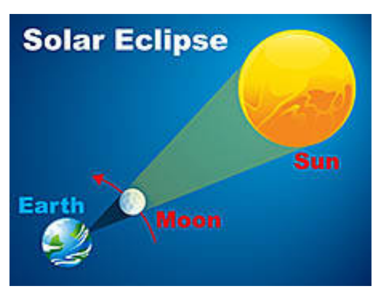
You have learned from Newton's 3rd Law that if the Earth exerts a gravitational force on the Moon, the Moon exerts an equal and opposite force on the Earth. But the force goes down with larger distances as $1/r^2$, so the part of the Earth closest to the Moon has a larger force exerted on it then the force exerted on the centre of the Earth. Similarly, the force exerted on the part of the Earth farthest from the Moon has a smaller force exerted on it.



For the water on the surface of the Earth, this causes the *tides*, which is the phenomenon that the height of the water in the ocean varies with time with a period of about 12 hours. The times of the high and low tides, and their height, can be calculated far into the future. For example in Sydney, Nova Scotia on Wednesday October 31, 2018, which is a year and a half after this document was written, there will be high tides at 1:30 AM and 2:26 PM. So we are using Newton's theory to predict the future.

Similarly, as you probably know, a solar eclipse is when the Moon is between the Sun and the Earth, and the shadow produced by the moon causes the eclipse. There will be a total solar eclipse in Toronto on Monday, April 8, 2024, which is almost 7 years in the future.

QUESTION 1. Why is the period of the tides about 12 hours instead of about 24 hours?



QUESTION 2. As shown in the figure for the tides on the previous page, the high tides are exactly on opposite sides of the earth. And the earth rotates on its axis with a period of almost exactly 24 hours. But the time between the high tides in Sydney, Nova Scotia on October 31, 2018 will be 12 hours and 54 minutes. Why isn't the time between the high tides almost exactly 12 hours?

This ability to predict the future is a key feature of classical physics. As Laplace wrote in 1814:

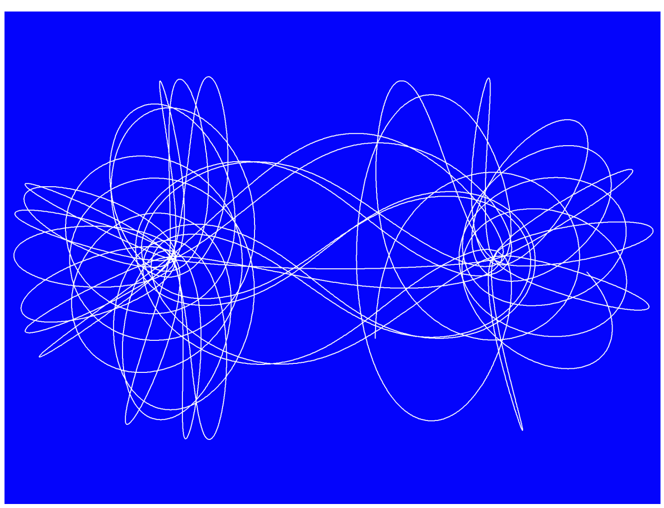
> We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

QUESTION 3. If Laplace is right, then do you have free will, i.e. do you have the ability to make a free choice about, say, whether to study physics or go to the movies tomorrow night?

III. The Three-Body Gravitational Problem

After Newton triumphed in solving the problem of a single fixed central mass with a single mass in orbit around it, he naturally decided to solve the second-most simple gravitational system: two fixed "Suns" with a single planet in orbit around them. He failed. Later many others tried to solve this *Three-Body Problem* and similarly failed.

In fact, it wasn't until about 1970 that a "solution" to this problem was found. The figure shows the result for a particular initial position of the planet for some finite time interval. Soon we will discuss why I put quote marks around the word *solution*.



I have prepared a Flash animation of this 3-body system with a slightly different initial position of the planet, which you may access at:

https://faraday.physics.utoronto.ca/PVB/Harrison/Flash/Chaos/ThreeBody/ThreeBody.html

I suggest at this point you just use the defaults of the animation, but soon you will be changing them. Run the animation.

Now, as promised, we explore why above I put the word *solution* in quote marks. If you look at the orbit of the planet, either in the figure on the previous page or in the Flash animation, then a true analytic solution to the problem is a formula that describes the orbit. A moment's reflection may convince you that this is not reasonable: if nothing else such a formula will be ghastly. It gets worse: we now know that the trajectory of the planet never ever repeats itself: after an infinite amount of time the length of the curve tracing the trajectory becomes infinitely long, but in a finite space. The conclusion, then, is that a "solution" to this problem, at least as we usually think of solutions in physics, does not exist.

Finding the "solution" to the three-body system involves a technique called *numerical integration*. The technique is only practical by using computers to do the calculations. Therefore, it was not until the 1960's when computers began to be available that we began to understand these systems, although before then some physicists, especially the great Henri Poincaré (1854 – 1912), managed to nibble around the edges of the problem. The Appendix describes how numerical integration is done.

Despite this issue with a lack of an analytic solution to the three-body gravitational problem, the system is deterministic: if we start the planet with exactly the same initial position and velocity, the trajectory will always be the same.

IV. Chaotic Systems

The three-body gravitational problem is an example of a *chaotic system*. The word *chaos* deserves a bit of explanation. In everyday life, the word chaos means a state of utter confusion or disorder, i.e. a total lack of organization or order. When physicists were first confronted with chaotic systems such as the 3-body gravitational one, the word was naturally adopted. As we learned more about these systems, we discovered that there are a number of characteristics and organizing principles that they all share. So the physics meaning of the word now is different than the everyday meaning. In our previous studies we have seen many examples of everyday words ending up with different meanings in physics, involving words like energy, momentum, etc.

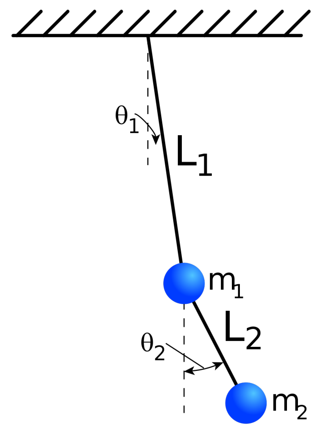
Get the Flash animation of the three-body system up in your browser again. Choose 4 independent planets using the control in the lower-right corner. Now there will be 4 non-interacting planets initially in exactly the same initial position. The *y* components of their initial velocities are slightly different: -1.00, -1.01, -1.02, and -1.03 in whatever units the animation is using. Now run the animation, and see what happens.

You will have seen that:

- Initially the trajectories of the 4 planets track each other pretty closely.
- After some time, the trajectories suddenly radically diverge.

Thus, we see that very small changes in the initial conditions lead to huge changes in the later behavior of the system. This is a characteristic of all chaotic systems, and is often called *Sensitive Dependence on Initial Conditions*. It is sometimes called the *butterfly effect* because if the climate is chaotic, then the formation of a tornado could depend on the flapping of the wings of a butterfly on the other side of the planet several weeks earlier.

There are many other systems that are chaotic. One is a double pendulum, such as is shown to the right. This system also exhibits the characteristics shared by all chaotic systems, including that fact that the motion of the balls never repeats, and the trajectories have Sensitive Dependence on Initial Conditions.

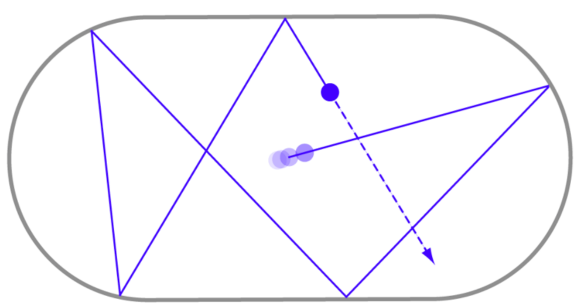


You may access an html5 animation of the double pendulum at:

http://www.tapdancinggoats.com/double-pendulum

Although you are, of course, free to explore the various options in this animation, just clicking on the *Run* button in the upper-left will show the essential features of the motion.

Another chaotic system is called the *Bunimovich Stadium*. The stadium has walls as shown and a particle collides with the walls in a perfectly elastic collision.

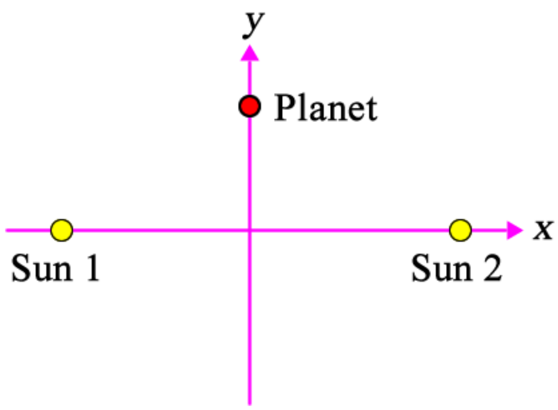


I have prepared a Flash animation of the Bunimovich Stadium, with two balls with identical initial velocities and slightly different initial positions. You may access the animation at:

https://faraday.physics.utoronto.ca/PVB/Harrison/Flash/Chaos/Bunimovich/Bunimovich.html

V. The Transition to Chaos

Not all initial configurations of a physical system are necessarily chaotic. For example, for the three-body gravitational system with fixed equal mass suns, if the initial position of the planet is on the perpendicular bisector of the line connecting the two suns, the *y* axis, and has no initial velocity in the horizontal direction, the motion of the planet will just be an oscillation up and down the *y* axis. There are other initial positions of the planet and masses of the sun that are also non-chaotic.



QUESTION 4. What are some initial conditions for the Bunimovich Stadium for which the particle will not exhibit chaotic motion?

For the three-body system with the mass as shown above, if we gradually move the initial horizontal position of the planet to the right, at some point the motion will become chaotic. The mathematics of exploring this transition to chaos is complicated, but we will use another system to show the essential features of the transition more simply. It is a very simple model of population dynamics called the *logistic map*.

Imagine we are trying to model the population of, say, rabbits in a forest. We know that, given what rabbits like to do, the increase in the population of rabbits will be related to the number of rabbits that we have. So we expect a term to look something like:

$$N_{\text{this generation}} = L \times N_{\text{previous generation}}$$

Here L is a constant representing the fecundity, i.e. the fertility etc., of the bunnies.

We also know that when there are too many rabbits in the forest, then the lack of food, overcrowding, etc. will suppress the number of rabbits in the next generation. If at a population of 100,000 all the rabbits die, then we need a term something like:

$$N_{\text{this generation}} = 100{,}000 - N_{\text{previous generation}}$$

Putting these two terms together gives us the *logistic equation*:

$$N_{\text{this generation}} = L \times N_{\text{previous generation}} \times \left(100{,}000 - N_{\text{previous generation}}\right) \qquad (2)$$

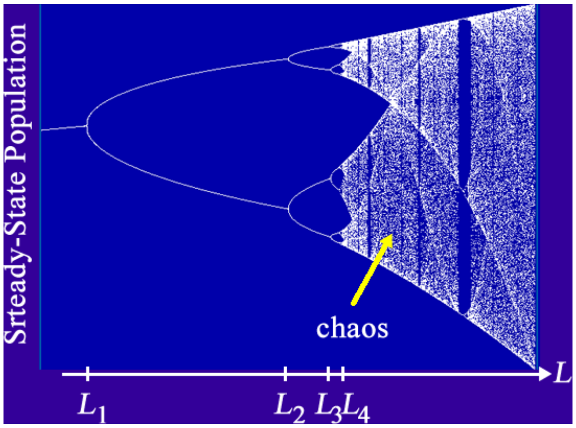
I have prepared a Flash animation solving the equation for various values of *L* which you may access at:

https://faraday.physics.utoronto.ca/PVB/Harrison/Flash/Chaos/LogisticMap/LogisticMap.html

I suggest you run the animation with the following values for *L*.

$L \leq 0.000\,029$	After some initial oscillations, the number of rabbits settles down to a steady-state value. As *L* increases, this steady-state value increases.
$0.000\,031 \leq L \leq 0.000\,033$	The steady-state number of rabbits oscillates between two values. We say the number of rabbits has *bifurcated*. As *L* increases from 0.000 031 to 0.000 033, the amplitude of the oscillation increases.
$L = 0.000\,035$	The steady-state number of rabbits has bifurcated again, and is oscillating between four different values.
$L = 0.000\,036$	The steady-state has bifurcated again, and is oscillating between eight different values.
$L > 0.000\,037$	The population of rabbits is now chaotic. This means that now the system exhibits all the properties shared by all chaotic systems, including the fact that the number of rabbits in any generation depends sensitively on the initial number of rabbits.

We can plot the steady-state values of the population as a function of the value of *L*. Here is the result.



As shown in the figure and as you have seen, for $L < L_1$ as L increases the steady-state number rabbits increases. The population bifurcates into two steady-state values at $L = L_1$, which then bifurcates into four steady-state values at $L = L_2$, which then bifurcates into eight steady-state values at $L = L_3$. Does that then bifurcate again into 16 steady-state values at some value $L = L_4$? It turns out that it does, and that in turn bifurcates once again into 32 steady-state values. And the bifurcation continues to an infinite number of steady-state values.

Here is a link to the YouTube video that illustrates:

https://www.youtube.com/watch?v=DcD74W2UcGg

This is another characteristic of all chaotic systems: **the transition to chaos is preceded by an infinite number of bifurcations.**

As shown in the video, the regions of bifurcations look the same regardless of the magnification of our view. This is called *self-similarity*.

Much of the early work on the logistic map was in the mid 1970's by Mitchell Feigenbaum while at the Los Alamos National Laboratory. At that time, computers were very expensive and he did not have access to one, but he had a Hewlett-Packard HP65 programmable calculator, which was very high-tech in its day but now would only be something in a museum. He was using his HP65 to calculate where the bifurcations in the logistic map would occur. These calculations took a long long time, and to amuse himself while waiting he started trying to predict where the next one would occur. Referring to the figure on the previous page, he discovered that:

$$\frac{L_2 - L_1}{L_3 - L_2} = \frac{L_3 - L_2}{L_4 - L_3} = \ldots = \frac{L_n - L_{n-1}}{L_{n+1} - L_n} \equiv \delta \cong 4.669 \tag{3}$$

δ is called the *Feigenbaum Constant*, and we now know that it is an irrational number whose value to 30 decimal places is 4.669201609102990671853203821578….

It turns out that for all chaotic systems the infinite bifurcations preceding the transition to chaos are similarly described by δ, so it is somehow involved in all chaotic systems. This is reminiscent of trying to work with circles: eventually you will need to come to terms with another irrational number, $\pi \cong 3.1415926$. Similarly, if you deal with a system where the change in some quantity is proportional to the value of the quantity, $dx \propto x$, you will soon end up dealing with the irrational number $e \cong 2.718282$. So there are three numbers, δ, π, and e, that have something to do with chaotic systems, circles, and exponential growth and decay respectively. Or perhaps these numbers are related to the way our minds think about these systems. In any case, how these arise is sort of mysterious.

VI. MORE ABOUT PREDICTING THE FUTURE

We have already pointed at that a chaotic system is strictly deterministic: if one starts the system from exactly the same initial conditions it will evolve in exactly the same way. This means that if we know the exact initial conditions and have access to a powerful enough computer, we can predict the exact state of the system at some later time.

This means that for the chaotic three-body gravitational system, if we know the exact initial position of the planet we can know its exact position at all later times.

But it is impossible to know the exact initial position of a real physical planet: any measurement of a real physical quantity such as the position of a planet is always uncertain. Perhaps you can determine the position of the planet to 2 decimal places, or to 20 decimal places, or even to 2000 decimal places, but you cannot determine the position of the planet to an infinite number of decimal places. Therefore Sensitive Dependence on Initial Conditions means that at some time in the future you will have no idea where the planet will be.

Earlier we mentioned that there will be a total eclipse of the sun in Toronto on Monday, April 8, 2024, which is almost 7 years in the future. But the trajectory of the Moon in orbit around the Earth is not just a simple ellipse: in addition to the gravitational force on it due to the Earth, there is a weaker gravitational force due to the Sun, and even weaker but non-zero forces due to the other planets, the other parts of our galaxy, the other galaxies, etc. If we just think of the Earth-Moon-Sun system, it is a three-body gravitational system and could be chaotic. So how can we be confident our prediction is of the future, when the solar eclipse occurs, is correct?

You can explore this further by once again getting the animation of the three-body system up in your browser. Set the controls to:

- 2 planets. This is in the lower-right corner of the animation.
- Mass of Sun 1 to 0. This is done by dragging the slider in the upper-left corner down.

Since the two planets do not interact, now it is just a simple system of a single Sun with a single planet. So the system is not chaotic and the motions of the planets are perfect ellipses with Sun 2 at one focus of the ellipse.

Run the animation.

You will have seen that the positions of the two planets differ somewhat, and the difference increases smoothly with time. This is a characteristic of non-chaotic systems: small differences in initial conditions lead to small differences in the time evolution of the system. So even if we don't know the initial position of a planet exactly, we can get a pretty good idea of how its trajectory evolves in time with only an approximate idea of the initial position.

Next keep the setting of 2 planets, but set the mass of Sun 1 back to its default value of 1 so the system is chaotic. Run the animation and roughly time how long it takes for the positions of the two planets radically diverge. Note that before the trajectories diverge, the system looks just like a non-chaotic one: the difference in positions increases smoothly with time.

Now keep the setting of 2 planets, but set the mass of Sun 1 to about 0.3 and run the animation, again timing how long it takes for the trajectories to radically diverge.

You will have seen that this system behaves in a non-chaotic fashion for a longer time then when the mass of Sun 1 was equal to 1.

So even for the chaotic three-body system, for some length of time it behaves just like a non-chaotic one: an approximate idea of the initial conditions can give an approximate result for the time evolution. The better our knowledge of the initial conditions, the better our knowledge of the later evolution.

VII. DISCUSSION

We have seen that all chaotic systems have the following properties:

- Sensitive Dependence on Initial Conditions
- The trajectory never repeats. Therefore, an analytical solution is not possible.
- The transition to chaos is preceded by infinite levels of bifurcation.
- The bifurcations are characterized by the Feigenbaum Constant.

There are many other characteristics shared by all chaotic systems. Thus we see that hidden in the apparent randomness of the trajectory of a chaotic system is a great deal of structure. One of the features of all chaotic systems which I reluctantly don't include here is that they all have fractional, i.e. non-integer, dimensionality.

Sensitive Dependence on Initial Conditions means that for a real physical system that is chaotic, since we cannot know the initial conditions perfectly a long-term prediction of its later evolution is impossible. However, for many such systems for short times an approximate prediction of its later state is possible.

For example, many people believe that the weather is a chaotic system. If they are correct, then a long-term weather forecast is impossible in principle.

We now know that chaotic systems are everywhere around us. We have seen examples in the three-body gravitational system, the double pendulum, the Bunimovich Stadium, and the logistic map model of population dynamics. Other examples include turbulent fluid flow, the arrhythmic beating of the heart, and epileptic brain activity.

There are at least more two issues regarding the ability to predict, say, when an eclipse will occur. One involves our understanding of the nature of gravitation, and the second is quantum mechanics.

The Nature of Gravitation

In the late 17th century Newton triumphed in explaining the motion of the Earth around the Sun and the Moon around the Earth with his 3 laws of motion and his theory of gravitation, Eqn. 1. This triumph not only explains what we observe, but also predicts the future position of the planets and Moon, although we need to add small perturbations because the systems are not just simple two-body ones with a central mass attracting a second one. The predictions were all experimentally verified within experimental uncertainties until the end of the 19th century when, with higher precision measurements, small discrepancies from the Newtonian description began to appear.

As you will learn next term, Einstein's General Theory of Relativity of 1915 is a better theory of gravitation, because it explains the small discrepancies from the Newtonian prediction. For example, in Einstein's theory the motion of a planet about a central fixed mass is still elliptical, but the axes of the ellipse precess, i.e. rotate. This has been experimentally observed for the orbit of Mercury. Here is a Flash animation that illustrates:

https://faraday.physics.utoronto.ca/PVB/Harrison/GenRel/Flash/Precession.html

Doing calculations with Einstein's theory is very difficult, so for less precise work we usually use Newton's instead. If necessary, we can correct the results by treating the precession as a small perturbation of the Newtonian solution.

Quantum Mechanics

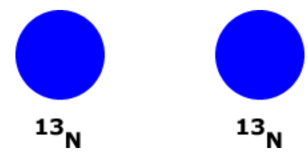
So far we have been exclusively discussing *classical* physics. The word *classical* means everything before the discovery of quantum mechanics in the mid-1920's. In classical physics, the universe is deterministic: identical initial conditions always leads to identical later states of the system. Chaotic systems are described by classical physics. In quantum mechanics this strict determinism is not true: identical initial conditions can lead to different later outcomes.

An example of a non-deterministic system is radioactivity. We characterize the tendency of a radioactive substance to decay by its *half-life*. If we have a large sample of radioactive atoms, in one half-life half of the atoms will have decayed and half will not. If we wait a further half-life, half of the remaining sample will decay and half will not. And so on. Here is a Flash animation illustrating for the made-up element Balonium with a half-life of 2 seconds.

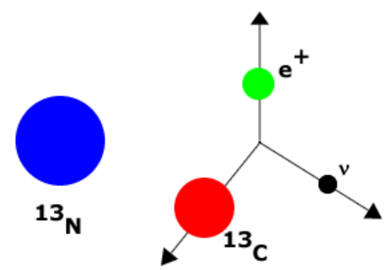
https://faraday.physics.utoronto.ca/PVB/Harrison/Flash/Nuclear/Decay/NuclearDecay.html

The decay of a radioactive atom is sort of like flipping a coin. For the coin, there is a 50% chance it will come up heads and a 50% chance it will come up tails, and whether the coin comes up heads or tails appears to be random. For a radioactive atom, after one half-life there is a 50% chance it will decay and a 50% chance it will not, and which will be the case similarly appears to be random.

The half-life of ^{13}N is almost exactly 10 minutes. Imagine that instead of a large number of ^{13}N atoms we have just two, as shown. We wait one half-life and ask: what happens? Perhaps both atoms have decayed, perhaps neither atom has decayed, and perhaps one has decayed and the other has not.



Imagine that the atom on the right decayed and the one on the left did not, as shown. We ask a basic question: **What is the difference between the two ^{13}N atoms?**



The answer is easy: one atom decayed and the other did not.

A more interesting question is: **What *was* the difference between the two atoms, before we waited 10 minutes?**

According to quantum mechanics, there was no difference between the two atoms. There is no mechanism inside the atom that determines whether or not it will decay after 10 minutes. There are no *hidden variables*.

Thus the world according to quantum mechanics is not strictly deterministic: identical initial conditions can lead to very different outcomes.

There is a difference between radioactive decay and a classical description of flipping a coin. If you know the initial conditions, the thrust, velocity, and rotation of the coin when it leaves your hand, the details of the surface on which the coin lands, etc. then at least in principle you can calculate whether it comes up heads or tails. Since you cannot control the exact way the coin leaves your hand, flipping a coin appears to be random, but it really isn't. Similarly the evolution of a chaotic system appears to be random, but is not.

Radioactive decay is believed to be truly random. As you may know, Einstein never accepted quantum mechanics, and this randomness is one of the reasons why. He said repeatedly, "God does not play dice with the universe." Bohr, one of the founders of quantum mechanics, replied, "Quit telling God what to do."

So far, physics has not been able to unify the General Theory of Relativity with quantum mechanics, although efforts are continuing.

TO LEARN MORE

The million-copy bestseller by James Gleick, *Chaos: Making a New Science* (Viking, 1987) is still a wonderful popular-level introduction. An enhanced e-book edition was released by Open Road Media in 2011, adding embedded video and hyperlinked notes.

There is also an excellent one-hour Nova program on chaos, available at:

https://www.youtube.com/watch?v=eJAs9Qr359o

We mentioned above that the infinite bifurcations preceding the transition to chaos exhibit self-similarity. This in turn means that chaotic systems are *fractals*. Benoit Mandelbrot pioneered the study of fractals, and the connection to chaos. His classic book *Fractals and Chaos: The Mandelbrot Set and Beyond* (Springer, 2004) is highly recommended.

In this department, PHY460 deals in part with chaotic systems. The textbook for the course is accessible to you with some effort: a particularly relevant version of the text is Steven H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (Perseus, 2015).

As implied by the "Nonlinear" in the title of Strogatz's book, chaotic systems are nonlinear, although not all nonlinear systems are chaotic. Stephen W. Morris, J. Tuzo Wilson Professor of Geophysics in this department, leads an Experimental Nonlinear Physics group, whose home page is: https://www.physics.utoronto.ca/~nonlin/. The site includes some amazing photographs of nonlinear systems.

APPENDIX

Here we discuss how to do a numerical integration to solve a system where analytic solutions do not exist. Our example will be the three-body gravitational system.

At some point in time t the planet has a position $\vec{r}(t)$, and velocity $\vec{v}(t)$. From $\vec{r}(t)$ we know the distance of the planet from Sun 1, r_1(t), and the distance from Sun 2, r_2(t). So, from Eqn. 1 we can calculate the force on the planet due to Sun 1, $\vec{F}_1(t)$ and the force due to Sun 2, $\vec{F}_2(t)$.

The total force on the planet is the vector sum of the two forces: $\vec{F}_{\text{tot}}(t) = \vec{F}_1(t) + \vec{F}_2(t)$.

At initial time $t = 0$ the initial position is $\vec{r}(0)$ and the total force on the planet is $\vec{F}_{tot}(0)$. Then the acceleration is:

$$\vec{a}(0) = \frac{\vec{F}_{\text{tot}}(0)}{m} \tag{A1}$$

where m is the mass of the planet. This is just Newton's 2nd Law.

Then we calculate the position of the planet at a small time Δt later:

$$\vec{r}(\Delta t) = \vec{r}(0) + \vec{v}(0) \times \Delta t \tag{A2}$$

You may already be saying to yourself, "Wait a minute! That is wrong." However, although you are right, if Δt is very small, it is very close to correct. In fact, we can make it as close to correct as we wish by making Δt small enough.

Similarly, we calculate the velocity of the planet at time Δt :

$$\vec{v}(\Delta t) = \vec{v}(0) + \vec{a}(0) \times \Delta t \tag{A3}$$

This too is wrong, but we can make it as close to correct as we wish by making Δt small enough.

We can calculate the force acting on the planet at time Δt since from Eqn. A2 we know its position at that time. Then Newton's 2nd Law lets us calculate the acceleration at time Δt .

From the values we found with Eqns. A2 and A3 and the value of the acceleration at time Δt , we can calculate the position and velocity at a time $2\Delta t$ with:

$$\begin{aligned} \vec{r}(2\Delta t) &= \vec{r}(\Delta t) + \vec{v}(\Delta t) \times \Delta t \\ \vec{v}(2\Delta t) &= \vec{v}(\Delta t) + \vec{a}(\Delta t) \times \Delta t \end{aligned} \tag{A4}$$

From this, we can calculate the acceleration at time $2\Delta t$ and then the position and velocity at a time $3\Delta t$ and so on. In general:

$$\begin{aligned} \vec{r}((n+1)\Delta t) &= \vec{r}(n\Delta t) + \vec{v}(n\Delta t) \times \Delta t \\ \vec{v}((n+1)\Delta t) &= \vec{v}(n\Delta t) + \vec{a}(n\Delta t) \times \Delta t \end{aligned} \tag{A5}$$

So, we can solve the three-body problem numerically, and if we have a powerful enough computer we can make Δt small enough that the approximations used are negligible. However, the number of calculations can be huge.

QUESTION A1. What is wrong with Eqn. A2 and Eqn. A3?

Question Answers

Here are the answers to the questions. As usual, I urge you to not look at them until you have given your best effort at answering them yourself, perhaps after discussion with your friends.

QUESTION 1. The Earth rotates with a period of 24 hours. So the bulge in the height of ocean closest to the Moon passes by a point of the rotating Earth every 24 hours. Similarly the bulge in the height of the ocean furthers from the Moon passes by a point of the Earth every 24 hours. So, the maximum height of the tides occurs twice every 24 hours. Therefore the period of the tides is about 12 hours.

QUESTION 2. Imagine an observer floating in free space a looking down at the earth and moon. The surface of the earth moves from west to east and a point on the equator will be moving at 1,670 km/hr relative to the observer. If the moon were stationary for this observer, the tides would be almost exactly 12 hours apart. But the moon is not stationary; it is orbit around the earth. It moves from west to east at 3,680 km/hr relative to this observer in outer space. So relative the an observer on earth, the moon moves to the east, and in 12 hours the moon, which defines where the high tides are, will have moved. This is also why a solar eclipse moves from west to the east.

QUESTION 3. If your consciousness is governed by classical physics, then free will is just an illusion. If your consciousness is not an example of classical physics, then perhaps you do have free will. As we will discuss later, quantum mechanics seems to imply the classical physics is just an approximation, and if your consciousness is a quantum effect then perhaps you have free will. The famous psychologist C.G. Jung was once asked, “Do we have free will or not?” He answered, “Yes.”

QUESTION 4. Here are two such initial conditions. If the particle is moving horizontally and is in the middle of the stadium it will just oscillate back and forth along the horizontal axis. If the particle is moving vertically and collides with the straight part of the wall of the stadium it will just oscillate up and down.

QUESTION A1. Eqn. A2 assumes that the velocity of the planet is constant between $t = 0$ and $t = \Delta t$. This is wrong, because the planet is accelerating. Eqn. A3 assumes that the acceleration of the planet is constant, which is also wrong: the distance of the planet from the suns is constantly changing so the acceleration is too. If you change Δt to a differential time dt, Eqns. A2 and A3 become correct.

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The figure of the Moon in orbit around the Sun on page 2 is from http://www.chegg.com/homework-help/questions-and-answers/common-thoughincorrect-statement-moon-orbits-earth-createsan-image-moon-s-orbit-looks-like-q440453 (Retrieved April 21, 2017).

The figure of the tides on page 2 is from http://sciencequestionswithsurprisinganswers.org/2012/12/14/i-know-that-the-gravity-of-the-moon-causes-ocean-tides-on-earth-how-does-centrifugal-force-cause-the-far-side-bulge/ although I have flipped the horizontal axis. (Retrieved April 22, 2017).

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